Two-Group Mixing Processes and Disease Dynamics in Single-Sex Populations

BU-1111-M

Jeffrey S. Palmer[1]
322 Warren Hall
Biometrics Unit
Cornell University
Ithaca, NY 14853-7801

Steven P. Blythe[2]
Dept. of Statistics &
Modelling Science
University of Strathclyde
Glasgow, Scotland
Great Britain

Carlos Castillo-Chavez
332 Warren Hall
Biometrics Unit/Center
for Applied Math/Pop. &
Development Program
Cornell University
Ithaca, NY 14853-7801

O. Abstract: The formulation of a mathematical description for social/sexual mixing among heterogeneous subgroups in a population has been a topic of intense investigation in recent years. Several solutions have been presented and the important role that mixing has on sexually transmitted disease epidemiology has been illustrated. Busenberg and Castillo-Chavez have developed a general representation which describes all solutions to the single-sex mixing problem. We offer a sociological interpretation which has enhanced our understanding of this representation. Additionally, we characterize a hypersurface upon which this representation reduces to proportionate mixing. For two interacting subgroups, the extremes in mixing behavior, complete assortative mixing and maximal disassortative mixing, are examined. A preference function is defined and used to illustrate the range of mixing behavior portrayed by various particular solutions to the mixing problem. The "worst-case" and "best-case" mixing strategies for each individual subgroup and for the population as a whole are described. Finally, conditions for stability of the disease-free steady-state of the population are calculated for both an SIS and SIR model and are shown to depend, in certain cases, on the mixing behavior of the population. Specific numerical simulations are given which illustrate these results.

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[1] To whom correspondence and information about reprints should be addressed. After June 15, 1991: College of Natural Sciences; Dakota State University; Madison, South Dakota 57042-1799

[2] Until August 1, 1991: 322 Warren Hall; Biometrics Unit; Cornell University; Ithaca, NY 14853-7801

I. **Introduction:** An understanding of the process, mathematical representation, and implications of social/sexual mixing among heterogeneous subgroups is crucial to our ability to understand the temporal dynamics of disease incidence and prevalence in populations. In recent years, spawned by efforts to model the HIV/AIDS epidemic, many solutions (some new and others rediscovered) of the mixing problem have been reported (REFS). Numerous investigators have illustrated the importance of sexual mixing patterns on the epidemiological modelling of sexually transmitted diseases (REFS). Of course, without appropriate data on sexual mixing patterns, the assumption of any particular mixing solution is strictly a matter of what an investigator believes to be reasonable for a given problem, and/or of exploration. Even so, analysis of the behavior of epidemiological models under various mixing assumptions has led to a better understanding of the role which mixing has on the development of model epidemics. This has both guided and stimulated the collection and statistical analysis of data on sexual mixing patterns (REFS).

In order to better understand the role of social/sexual mixing on disease dynamics we must first clarify the various mixing patterns which can occur between population subgroups. Only when we have a clear understanding of the mixing behavior and how it changes as subgroup size changes or how it changes through time can we begin to assess the implications of mixing on the dynamics of an epidemic. This is true in both theoretical modelling studies and in practical applications. Armed with a clear picture of the mixing pattern of the population we may be able to better explain the resulting spread of the disease. For instance, as a subgroup tends to decrease in size relative to the other subgroups, does their mixing become more assortative or more disassortative. Clearly, depending on the relative prevalence of disease in the various subgroups, this change in mixing pattern could have important implications for the future dynamics of the disease in the population. Thus, when a particular mixing solution is incorporated into an epidemic model, it is important to first understand the mixing assumptions which are being made and then to interpret the model results in light of those assumptions.

Using an axiomatic approach, Busenberg and Castillo-Chavez obtained a general representation for all solutions to both the single-sex (REFS) and two-sex (REFS) mixing problems in subdivided populations with or without age-structure. More recently, Blythe (REF) has extended this result (without age-structure) to allow for arbitrary subgroup connectedness. All other solutions to the mixing problem can be recovered as special cases of these general representations. This has been explicitly demonstrated for all of the mixing solutions which hav: thus far appeared in the literature (REFS). Therefore, we refer to this representation as general mixing.

Only through a better understanding of the mixing behavior of the population can we better understand the role of social/sexual mixing on the development of sexually transmitted disease epidemics. Toward this end, we offer a sociologically intuitive explanation for general mixing among heterogeneous subgroups in a single-sex population. A potentially useful new idea, allowing the parameters in the general representation to take on negative values, is introduced and is shown to be consistent with this interpretation. Secondly, we characterize a hypersurface upon which the general representation reduces to proportionate mixing. For two interacting subgroups, two mixing extremes, complete assortative mixing and maximal disassortative mixing are described. Preference is defined and used to examine the range of mixing behaviors covered by some previously investigated mixing solutions. So-called "worst-case" and "best-case" mixing strategies, for each subgroup in the population and for the population as a whole, are considered. Finally, we analytically examine the role that social/sexual mixing can have on the stability of the disease-free equilibrium of a population in both an SIS and SIR epidemic model. Specific examples of these results are illustrated. In conclusion, we discuss the extension of these results to three or more subgroups as well as to the modelling of more general sexually interacting populations.

II. General Mixing in a Single-Sex Population: We begin with a description of the social/sexual mixing problem for a single-sex population, all intragroup and intergroup mixing pathways are open, partitioned into N distinct subgroups. An example, with N=2 , is depicted in Figure 1.

Let $T_j(t)$ be the number of sexually active individuals in the j^{th} subgroup at time t, and let $\alpha_j(t)$ denote the average number of sexual partnerships which an individual in subgroup j engages in per unit time. Then, if $p_{ij}(t)$ is the fraction of subgroup i's partnerships which occur with individuals from subgroup j, the mixing problem is to define $p_{ij}(t)$ subject to the four constraints (REFS):

$$0 \leq p_{ij}(t) \leq 1 \tag{i}$$

$$\sum_{j=1}^{N} p_{ij}(t) = 1 \tag{ii}$$

$$\alpha_i(t)T_i(t)p_{ij}(t) = \alpha_j(t)T_j(t)p_{ji}(t) \tag{iii}$$

and

$$\alpha_i(t)T_i(t) = 0 \quad \Rightarrow \quad p_{ji}(t) = 0, \quad \forall\, j \neq i. \tag{iv}$$

Constraints (i) and (ii) make $p_{ij}(t)$ a stochastic matrix and ensure that each subgroup exactly attains its targeted quota of sexual partnerships. Clearly, a sexual partnership between an individual in subgroup i and an individual in subgroup j implies the simultaneous formation of a sexual partnership between an individual in subgroup j and an individual in subgroup i. Constraint (iii) is a group reversibility property which reflects this obvious biological fact. Constraint (iv) implies that if the i^{th} subgroup is either empty, $T_i(t) = 0$, or sexually inactive, $\alpha_i(t) = 0$, then all other subgroups may not obtain partners from subgroup i. Note that the particular values in the i^{th} row of $p_{ij}(t)$ do not matter since the force of infection term for the i^{th} subgroup (see section IV),

$$B_i(t) = \alpha_i(t)\beta S_i(t)\sum_{j=1}^{N} p_{ij}(t)\,\frac{I_j(t)}{T_j(t)}\,, \tag{1}$$

vanishes independent of the particular values of $p_{ij}(t)$. For $\alpha_i(t)T_i(t) = 0$, the values in the i^{th} row of $p_{ij}(t)$ may be interpreted as the limiting value of the mixing behavior of subgroup i as $\alpha_i(t)T_i(t)$ approaches zero. In other words, this describes the mixing behavior of subgroup i as it becomes very small (with respect to the total supply of sexual partnerships in the population) compared to the other subgroups.

Busenberg and Castillo-Chavez (REFS) have demonstrated that all solutions to the mixing problem described by (i) - (iv) may be represented in the form

$$p_{ij}(t) = \bar{p}_j(t)\left[\frac{R_i(t)R_j(t)}{\sum_{k=1}^{N}\bar{p}_k(t)R_k(t)} + \phi_{ij}(t)\right] \tag{2}$$

where

$$\bar{p}_j(t) = \frac{\alpha_j(t)T_j(t)}{\sum_{k=1}^{N}\alpha_k(t)T_k(t)}\,, \tag{3}$$

$$R_i(t) = 1 - \sum_{k=1}^{N}\phi_{ik}(t)\bar{p}_k(t)\,, \tag{4}$$

and $\phi(t)$ is a symmetric, nonnegative, NxN matrix such that $R_i(t) \geq 0$ for all i and $R_i(t) > 0$ for at least one i. Here, $\bar{p}_j(t)$ represents proportionate mixing, the solution obtained when sexual partnerships are distributed in accordance with the relative availability in each subgroup. The general representation (2) is a multiplicative perturbation of proportionate mixing.

There have been other approaches to the formulation of a mathematical solution to the mixing problem. A number of particular solutions, including assortative mixing (REFS), proportionate mixing (REFS), preferred mixing (REFS), like-with-like mixing (REFS), biased mixing (REFS), and structured mixing (REFS), seem to have been derived from the specification of a sociological mechanism which then generated a mathematical formulation of the problems solution. Others have chosen to avoid the explicit solution of the mixing problem, incorporating instead into their models assumptions which either maintain a constant population size (REFS) or, alternatively, they choose a fixed p_{ij} and allow sexual activity rates, $\alpha_i(t)$ to change (REFS). The novel axiomatic approach taken by Busenberg and Castillo-Chavez in the development of the general representation (2) has, however, been somewhat overshadowed by the apparent lack of a sociological interpretation; below, we offer an interpretation for this representation.

This idea is presented for N interacting subgroups and is illustrated, in Figure 1, for N=2. On average, an individual from the i^{th} subgroup must obtain $\alpha_i(t)$ sexual partnerships per unit time. The general representation (2) in conjunction with axiom (i) imply that

$$p_{ij}(t) \geq \bar{p}_j(t)\phi_{ij}(t) . \tag{5}$$

Thus, on average, an individual in subgroup i forms partnerships with individuals from subgroup j at a minimum rate of

$$\alpha_i(t)\phi_{ij}(t)\bar{p}_j(t) . \tag{6}$$

Consequently, summing over all subgroups, we conclude that an average individual in the i^{th} subgroup has formed

$$\alpha_i(t)\sum_{k=1}^{N}\phi_{ik}(t)\bar{p}_k(t) = \alpha_i(t)[1 - R_i(t)] \tag{7}$$

sexual partnerships per unit time. Hence, there are $\alpha_i(t)R_i(t)$ additional partnerships remaining to be formed during one unit of time.

Now, consider a population divided into N distinct subgroups, each of size $T_i(t)$ as before, but with a sexual activity rate of $\alpha_i(t)R_i(t)$ per individual per unit time. Assume proportionate mixing among the subgroups in this "new" population. Then, an average individual from subgroup i forms partnerships with individuals from subgroup j at the rate of

$$\alpha_i(t)R_i(t)\frac{\alpha_j(t)R_j(t)T_j(t)}{\sum_{k=1}^{N}\alpha_k(t)T_k(t)R_k(t)} = \frac{\alpha_i(t)R_i(t)R_j(t)\overline{p}_j(t)}{\sum_{k=1}^{N}\overline{p}_k(t)R_k(t)} \tag{8}$$

per unit time. Adding together the results in (6) and (8) above, we see that a typical individual from subgroup i has now formed

$$\alpha_i(t)\left[\phi_{ij}(t)\overline{p}_j(t) + \frac{R_i(t)R_j(t)\overline{p}_j(t)}{\sum_{k=1}^{N}\overline{p}_k(t)R_k(t)}\right] = \alpha_i(t)p_{ij}(t) \tag{9}$$

partnerships per unit time with individuals from subgroup j. Clearly, $p_{ij}(t)$ is given by (2), the general mixing representation.

General mixing may be thought of as reserving a fraction, $\phi_{ij}(t)\overline{p}_j(t)$, of subgroup i's rate of pair formation for interactions with subgroup j. This fraction is the product of $\phi_{ij}(t)$ and $\overline{p}_j(t)$, a measure of the relative availability of partners. The remaining rate of pair formation is distributed at random among the various subgroups as if they were forming partnerships at a reduced rate of $\alpha_i(t)R_i(t)$. Since all mixing solutions can be represented by general mixing, this provides an explanation of the way in which mixing between subgroups of a single-sex can be described. However, this does not imply that mixing actually occurs in this way among individuals, a problem for which explicit pair-formation models are more appropriate.

Several solutions to the single-sex mixing problem, among them, assortative, proportionate, and preferred mixing, have been used in the mathematical modelling of sexually transmitted disease epidemics. Blythe and Castillo-Chavez (REFS) have explicitly shown that each of these may be written in the form of general mixing by making an appropriate choice for $\phi_{ij}(t)$. The only constraints are that each $R_i(t)$ be nonnegative, that at least one $R_i(t)$ be positive, and that $\phi(t)$ be symmetric (REFS). A sufficient condition is given by $0 \le \phi_{ij}(t) \le 1$, and a wide range of mixing patterns may be modelled with constant ϕ_{ij} from this interval.

Clearly, $0 \le \phi_{ij}(t) \le 1$, is not a necessary condition for the specification of a mixing solution. Mathematically valid and sociologically meaningful solutions may be generated by ϕ_{ij}'s which are not restricted to the unit interval. Preferred mixing (REFS),

$$p_{ij}(t) = \delta_{ij}a_i + (1-a_i)\frac{(1-a_j)\overline{p}_j(t)}{\sum_{k=1}^{N}(1-a_k)\overline{p}_k(t)} \tag{10}$$

may be equivalently written (REFS) using the general representation (2) by taking

$$\phi_{ij}(t) = \frac{\delta_{ij}\, a_i}{\overline{p}_i(t)} \tag{11}$$

where $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and $0 \leq a_i \leq 1$. Preferred mixing may be interpreted as reserving a fraction, a_i, of subgroup i's sexual partnerships for intragroup mixing and distributing all remaining partnerships proportionately among the various subgroups. Our interpretation of the general representation agrees with this interpretation of preferred mixing. Clearly, the $\phi_{ij}(t)$ given by (11) above satisfy the necessary constraints on the general representation, however, the $\phi_{ii}(t)$'s are neither constant or are they restricted to the unit interval. This is an example where $\phi_{ij}(t)$ is "frequency-dependent" (i.e. $\phi_{ij}(t)$ is a function of the $\overline{p}_j(t)$'s).

All mixing solutions may be obtained from the general representation (2) where each $\phi_{ij}(t)$ is strictly non-negative (REFS). However, examples of valid mixing solutions can be generated using the general representation (2) and negative ϕ_{ij}'s. For example, if we take

$$\phi = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}. \tag{12}$$

and use the general representation (2) to calculate p_{ij} we obtain

$$p_{ij} = \begin{bmatrix} 1 - \dfrac{(1+a)\overline{p}_2}{2} & \dfrac{(1+a)\overline{p}_2}{2} \\ \dfrac{(1+a)\overline{p}_1}{2} & 1 - \dfrac{(1+a)\overline{p}_1}{2} \end{bmatrix}. \tag{13}$$

It is easy to check that (13) satisfies axioms (i)-(iv) for $-1 \leq a \leq 1$. We remark that this is an alternative representation of preferred mixing in the special case where each subgroup reserves the same fraction of its partnerships for within group mixing. Another example is obtained by choosing

$$\phi = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}. \tag{14}$$

Then, the mixing solution becomes

$$p_{ij} = \begin{bmatrix} 1 - \dfrac{(1-a^2\bar{p}_1\bar{p}_2)\bar{p}_2}{1-2a\bar{p}_1\bar{p}_2} & \dfrac{(1-a^2\bar{p}_1\bar{p}_2)\bar{p}_2}{1-2a\bar{p}_1\bar{p}_2} \\ \\ \dfrac{(1-a^2\bar{p}_1\bar{p}_2)\bar{p}_1}{1-2a\bar{p}_1\bar{p}_2} & 1 - \dfrac{(1-a^2\bar{p}_1\bar{p}_2)\bar{p}_1}{1-2a\bar{p}_1\bar{p}_2} \end{bmatrix} \tag{15}$$

which satisfies axioms (i)-(iv) for $-2 \leq a \leq 2$. An advantage of allowing negative entries in ϕ is that it allows us to represent a wider range of mixing solutions with constant parameters. This could prove to be a useful advantage when trying to estimate the ϕ_{ij}'s from survey data. Since the formation of partnerships at a negative rate is biologically unreasonable, we show that, properly interpreted, this is consistent with our sociological interpretation of the general representation. Consider the case

$$\phi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{16}$$

and the population illustrated in Figure 2. Then $\alpha_j\phi_{ij}\bar{p}_j(t)$ (perhaps negative) partnerships are reserved along each connection. The effective subgroup population size, $\alpha_j R_j(t)T_j(t)$, for proportionate mixing is now larger than it would have been if the ϕ_{ij}'s were strictly nonnegative. Making the proportionate mixing assignments, and adding together the two steps we obtain an example of a valid mixing solution (complete assortative mixing). An accounting of the "negative partnerships" finds them redistributed proportionately among the other subgroups in the population.

II. Proportionate Mixing Solutions: We now describe a hypersurface upon which the general mixing representation (2) reduces to proportionate mixing. Let $G_i(t)$ be nonnegative and finite, then

$$\phi_{ij}(t) = 1 - G_i(t)G_j(t) \tag{17}$$

describes a hypersurface in ϕ space and as long as $\phi_{ij}(t)$ remains on this surface there will be proportionate mixing between subgroups i and j. For N subgroups there are N(N-1)/2 such surfaces defined. If $\phi_{ij}(t)$ is on the intersection of all of these surfaces, $p_{ij}(t)$ reduces to proportionate mixing. These surfaces always have a nontrivial intersection; if $\phi_{ij}(t) \equiv c$ for all i,j then (2) reduces to proportionate mixing. In this case, the preferences of various subgroups are all identical; that this results in proportionate mixing is not surprising. However, this is not the only case which gives rise to proportionate mixing as is clear from (17). If we consider two interacting subgroups and let

$$\phi = \begin{bmatrix} A & B \\ B & C \end{bmatrix} \tag{18}$$

where A, B, and C are constant, then

$$\det\phi - \mathrm{tr}\phi + 2B = AC - B^2 - A + 2B - C = 0 \tag{19}$$

describes a proportionate mixing surface. This surface is plotted in Figure 3 and represents all possible cases of proportionate mixing between two interacting subgroups. While it contains the line A = B = C, it also contains an infinite number of other proportionate mixing solutions as well. In this case, even though there is definite preferential affinity between subgroups the net result is proportionate mixing. We remark that this non-unique representation of proportionate mixing is not unreasonable, in fact, it has been regularly observed in the population genetics literature (REFS).

For N=3 and

$$\phi = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \tag{20}$$

we derive from (17) the following three surfaces for proportionate mixing:

Surface	Proportionate Mixing Subgroups	
$A - AD + B^2 - 2B + D = 0$	1 & 2	(21)
$A - AF + C^2 - 2C + F = 0$	1 & 3	(22)
$D - DF + E^2 - 2E + F = 0$	2 & 3	(23)

Finally, we note that the existence of a proportionate mixing surface between any two subgroups divides the mixing behavior between those two subgroups into two categories. On one side of this surface mixing is more assortative than proportionate mixing and on the other side mixing is more disassortative than proportionate mixing.

III. Social/Sexual Mixing Between Two Interacting Subgroups: For a single-sex population subdivided into two interacting subgroups there are two clearly defined extremes in mixing behavior: complete assortative mixing and maximal disassortative mixing. The trivial solution, complete assortative mixing, is given by

$$p_{ij}(t) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \tag{24}$$

in this case there is no mixing between the two subgroups in the population. This is a special case of preferred mixing (10), where each $a_i \equiv 1$ and may be recovered from the general representation (2) by taking

$$\phi = \begin{bmatrix} \frac{1}{\bar{p}_1} & 0 \\ 0 & \frac{1}{\bar{p}_2} \end{bmatrix}. \tag{25}$$

The other extreme in mixing behavior, maximal disassortative mixing, is given by

$$p_{ij}(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ \frac{\bar{p}_1}{\bar{p}_2} & 1-\frac{\bar{p}_1}{\bar{p}_2} \end{bmatrix} & \text{if } \bar{p}_1 \le \bar{p}_2 \\ \begin{bmatrix} 1-\frac{\bar{p}_2}{\bar{p}_1} & \frac{\bar{p}_2}{\bar{p}_1} \\ 1 & 0 \end{bmatrix} & \text{if } \bar{p}_1 \ge \bar{p}_2 \end{cases} \tag{26}$$

and may be recovered from the general representation (2) by taking

$$\phi = \begin{bmatrix} 1 & \min\{\frac{1}{\bar{p}_1}, \frac{1}{\bar{p}_2}\} \\ \min\{\frac{1}{\bar{p}_1}, \frac{1}{\bar{p}_2}\} & 1 \end{bmatrix}. \tag{27}$$

In this case, the subgroup with the smaller value of $\alpha_i(t)T_i(t)$ obtains all of its sexual partnerships from the other subgroup.

If ϕ_{ij} is not explicitly time-dependent, then p_{ij} only depends explicitly on $\bar{p}_1$ ($\bar{p}_2 = 1-\bar{p}_1$), which changes with time according to the dynamical equations of the governing model. Hence, we can explore mixing, p_{ij} as a function of $\bar{p}_1$ independent of the governing model equations. Our goal in what follows is to explore the range of mixing behavior covered by various solutions of the mixing problem. We have already seen that different choices for the matrix ϕ can generate the same mixing solutions. Thus, we define the "preference function"

$$\psi(\bar{p}_1) = \frac{p_{12}}{1 - \bar{p}_1} = \frac{p_{21}}{\bar{p}_1} = \frac{R_1 R_2}{\bar{p}_1 R_1 + \bar{p}_2 R_2} + \phi_{12} \tag{28}$$

which gives a measure of the density-dependent effects on mixing between the two subgroups. The "preference function" is the ratio of the number of partnerships shared between the two subgroups to the number that they would share under proportionate mixing. For complete assortative mixing,

$$\psi(\bar{p}_1) \equiv 0\,, \tag{29}$$

for maximal disassortative mixing

$$\psi(\bar{p}_1) = \min\{\frac{1}{\bar{p}_1}, \frac{1}{\bar{p}_2}\}\,, \tag{30}$$

and for proportionate mixing

$$\psi(\bar{p}_1) \equiv 1\,; \tag{31}$$

each of these cases is illustrated in Figure 4. The preference function associated with any other mixing solution must lie between the curves defined by (29) and (30), thus

$\psi(\overline{p}_1)$ allows us to graphically illustrate the behavior of any mixing solution relative to these two extremes and proportionate mixing. Furthermore, any two mixing solutions with the same preference function are identical solutions. Thus, in this framework, each mixing solution is uniquely represented by its preference function.

For example, the preference function for preferred mixing

$$\psi(\overline{p}_1) = \frac{(1\text{-}a_1)(1\text{-}a_2)}{(1\text{-}a_1)\overline{p}_1 + (1\text{-}a_2)\overline{p}_2} \tag{32}$$

preferred mixing always lies between that of complete assortative mixing and proportionate mixing; preferred mixing is always more assortative than is proportionate mixing. Several examples are plotted in Figure *. If $a_1 > a_2$ the preference function is monotone increasing; if $a_2 > a_1$ it is monotone decreasing and if $a_1 = a_2$ the preference function is constant. Also note that $\psi(0) = 1\text{-}a_1$ and $\psi(1) = 1\text{-}a_2$; thus, as either subgroup becomes "small" with respect to the other, the mixing does not necessarily approach proportionate mixing.

The general representation, where each ϕ_{ij} is a constant from the unit interval has an assortative mixing extreme when

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{33}$$

and a disassortative mixing extreme when

$$\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{34}$$

The preference functions corresponding to each of these extremes are shown in Figure 6. Consider now ϕ given by (18). For the single parameter submodel A=B=C we always obtain proportionate mixing. We also obtain proportionate mixing in the two parameter submodels where A=B or B=C. In each of these cases the preference function is given by (31) above. Examples of the preference functions associated with the two-parameter submodel A=C are shown in Figure 6. In this case, preference is always symmetric about $\overline{p}_1 = 1/2$, and is assortative and convex if A>B, and is disassortative and concave if A<B. For the full three parameter model where A, B, and C are all distinct, the

mixing pattern is asymmetric, assortative and convex if AC-B^2-A-C+2B is negative, and is disassortative and concave if AC-B^2-A-C+2B is positive. Some examples are shown in Figure 6. To examine the mixing behavior of "small" subgroups we notice that $\psi(0)=\psi(1)=1$, proportionate mixing, unless $\phi_{11}=1$ and/or $\phi_{22}=1$.

IV. Mixing Strategies for Two Interacting Subgroups: In this section we introduce a model for the spread of a sexually transmitted disease in a single-sex population with two interacting subgroups and examine the "worst-case" and "best-case" mixing strategies for each individual subgroup and for the population as a whole.

Let $S_i(t)$ represent the number of susceptible individuals and let $I_i(t)$ be the number of infected (and assumed infectious) individuals in the i^{th} subgroup at time t respectively. We assume that recruitment occurs only in the susceptible classes and at a constant rate Λ_i. Define α_i as the sexual activity rate (assumed constant) of the i^{th} subgroup (partners/individual/year) and assume that the transmissivity rate is given by β_{ij}. Let μ be the removal rate from each susceptible class and let δ be the rate of removal from the infected classes. Finally, let r be the rate at which infected individuals recover and return to the susceptible class and let p_{ij} be any mixing solution which satisfies axioms (i)-(iv). Using these definitions we obtain the following set of ordinary differential equations describing the spread of disease in this population:

$$\frac{dS}{dt} = \Lambda_i - \alpha_i S_i \sum_{j=1}^{2} p_{ij} \frac{I_j}{S_j+I_j}\beta_{ij} - \mu S_i + rI_i \tag{35}$$

$$\frac{dI}{dt} = \alpha_i S_i \sum_{j=1}^{2} p_{ij} \frac{I_j}{S_j+I_j}\beta_{ij} - (\delta + r)\, I_i\,. \tag{36}$$

If $\delta = \mu$, then (35) and (36) reduce to an SIS model and for $r = 0$ the system represents an SIR model.

The force of infection in subgroup 1 is given by

$$B_1 = \alpha_1 S_1 (p_{11}\mathfrak{I}_1\beta_{11} + p_{12}\mathfrak{I}_2\beta_{12}) \qquad \text{where } \mathfrak{I}_i = \frac{I_i}{T_i}\text{, and } T_i = S_i + I_i\,. \tag{37}$$

We wish to choose p_{11} and p_{12} so that B_1 is respectively maximized (minimized) at a given time t. Recalling that $p_{12} = 1 - p_{11}$, we obtain:

$$B_1^{max} \text{ occurs under} \begin{cases} \text{complete assortative mixing} & \text{if } \mathfrak{I}_1\beta_{11} > \mathfrak{I}_2\beta_{12} \\ \text{maximal disassortative mixing} & \text{if } \mathfrak{I}_1\beta_{11} < \mathfrak{I}_2\beta_{12} \end{cases} \tag{38}$$

$$\text{complete assortative mixing} \quad \text{if } \mathfrak{I}_1\beta_{11} < \mathfrak{I}_2\beta_{12}$$

$$B_1^{\min} \text{ occurs under } \begin{cases} \\ \text{complete disassortative mixing if } \mathfrak{s}_1\beta_{11} > \mathfrak{s}_2\beta_{12} \end{cases} \tag{39}$$

Maximizing (minimizing) the force of infection in one subgroup minimizes (maximizes) the force of infection in the other group. Thus, the "worst case" mixing scenario for one subgroup is the "best case" scenario for the other subgroup.

The total force of infection in the population is given by:

$$\begin{aligned} B_T &= B_1 + B_2 \\ &= \alpha_1 S_1 \left[p_{11}\mathfrak{s}_1\beta_{11} + p_{12}\mathfrak{s}_2\beta_{12}\right] + \alpha_2 S_2 \left[p_{21}\mathfrak{s}_1\beta_{21} + p_{22}\mathfrak{s}_2\beta_{22}\right]. \end{aligned} \tag{40}$$

Recall that $\alpha_1 T_1 p_{12} = \alpha_2 T_2 p_{21}$, so

$$p_{21} = \frac{\alpha_1 T_1}{\alpha_2 T_2} p_{12} = \frac{\bar{p}_1}{\bar{p}_2} p_{12} \qquad \text{and} \qquad p_{22} = 1 - p_{21} = 1 - \frac{\bar{p}_1}{\bar{p}_2} p_{12}. \tag{41}$$

Noting that at a fixed time t, B_T is of the form

$$\begin{aligned} B_T &= ap_{11} + bp_{12} + cp_{21} + dp_{22} \\ &= a(1 - p_{12}) + bp_{12} + c\left(\frac{\bar{p}_1}{\bar{p}_2} p_{12}\right) + d\left(1 - \frac{\bar{p}_1}{\bar{p}_2} p_{12}\right) \\ &= \left[-a + b + c\frac{\bar{p}_1}{\bar{p}_2} - d\frac{\bar{p}_1}{\bar{p}_2}\right] p_{12} + [a + d], \end{aligned} \tag{42}$$

we see that B_T has a minimum (maximum) at $p_{12} = 0$, complete assortative mixing, when

$$-a + b + c\frac{\bar{p}_1}{\bar{p}_2} - d\frac{\bar{p}_1}{\bar{p}_2} > (<)\ 0 \tag{43}$$

respectively. The minimum (maximum) occurs at $p_{12} = 1$, maximal disassortative mixing, when

$$-a + b + c\frac{\bar{p}_1}{\bar{p}_2} - d\frac{\bar{p}_1}{\bar{p}_2} < (>)\ 0 \tag{44}$$

respectively. Thus the sign of

$$-a+b+c\frac{\bar{p}_1}{\bar{p}_2}-d\frac{\bar{p}_1}{\bar{p}_2} \tag{45}$$

or equivalently, the sign of

$$\begin{aligned} Q(t) &\equiv \left[\alpha_1 S_1 \mathfrak{I}_2 \beta_{12} - \alpha_1 S_1 \mathfrak{I}_1 \beta_{11}\right]\bar{p}_2 + \left[\alpha_2 S_2 \mathfrak{I}_1 \beta_{21} - \alpha_2 S_2 \mathfrak{I}_2 \beta_{22}\right]\bar{p}_1 \\ &= \alpha_1 S_1\left[\mathfrak{I}_2 \beta_{12} - \mathfrak{I}_1 \beta_{11}\right]\bar{p}_2 + \alpha_2 S_2\left[\mathfrak{I}_1 \beta_{21} - \mathfrak{I}_2 \beta_{22}\right]\bar{p}_1 \end{aligned} \tag{46}$$

determines the mixing pattern (either complete assortative mixing or maximal disassortative mixing) which gives the extremes in B_T, the total force of infection in the population. If $Q(t) > 0$, B_T has a maximum under maximal disassortative mixing and a minimum under complete assortative mixing. When $Q(t) < 0$ the maximum of B_T occurs under complete assortative mixing is in place and the minimum under maximal disassortative mixing is operating.

V. Stability of the Disease-Free State: We now develop conditions for stability of the disease-free equilibrium of the model given by (35) and (36) when $p_{ij}(t)$ is given by the general mixing representation (2) and ϕ_{ij} is constant. Furthermore, we assume that $\beta_{ij} \equiv \beta$.

Recalling that $T_i(t) = S_i(t) + I_i(t)$, equations (35) and (36) may be equivalently rewritten as:

$$\frac{dT_1}{dt} = \Lambda_1 - \mu(T_1 - I_1) - \delta I_1 \tag{47}$$

$$\frac{dI_1}{dt} = \alpha_1(T_1 - I_1)\beta[\, p_{11}\frac{I_1}{T_1} + p_{12}\frac{I_2}{T_2}\,] - (\delta + r)I_1 \tag{48}$$

$$\frac{dT_2}{dt} = \Lambda_2 - \mu(T_2 - I_2) - \delta I_2 \tag{49}$$

$$\frac{dI_2}{dt} = \alpha_2(T_2 - I_2)\beta[\, p_{21}\frac{I_1}{T_1} + p_{22}\frac{I_2}{T_2}\,] - (\delta + r)I_2 \tag{50}$$

We wish to calculate the stability of the disease-free equilibrium,

$$(\, T_1, I_1, T_2, I_2 \,) \;=\; (\, \frac{\Lambda_1}{\mu}, 0, \frac{\Lambda_2}{\mu}, 0 \,) \tag{51}$$

when p_{ij} is given by the general representation (2) of Busenberg and Castillo-Chavez and each ϕ_{ij} is assumed to be constant. The Jacobian of (47)-(50) evaluated at the disease-free equilibrium is

$$J = \begin{bmatrix} -\mu & \mu-\delta & 0 & 0 \\ 0 & \alpha_1\beta\overset{*}{p}_{11} - (\delta + r) & 0 & \dfrac{\alpha_1\beta\Lambda_1\overset{*}{p}_{12}}{\Lambda_2} \\ 0 & 0 & -\mu & \mu-\delta \\ 0 & \dfrac{\alpha_2\beta\Lambda_2\overset{*}{p}_{21}}{\Lambda_1} & 0 & \alpha_2\beta\overset{*}{p}_{22} - (\delta + r) \end{bmatrix}, \qquad (52)$$

where $\overset{*}{p}_{ij}$ is the value of p_{ij} evaluated at the disease-free equilibrium. The characteristic equation is found by setting

$$\det(J-\lambda I) = 0 \qquad (53)$$

yielding

$$(\lambda+\mu)^2[\lambda^2-(\alpha_1\beta\overset{*}{p}_{11}+\alpha_2\beta\overset{*}{p}_{22}-2\delta-2r)\lambda+(\alpha_1\beta\overset{*}{p}_{11}-\delta-r)(\alpha_2\beta\overset{*}{p}_{22}-\delta-r)-\alpha_1\alpha_2\beta^2\overset{*}{p}_{12}\overset{*}{p}_{21}]=0 \qquad (54)$$

For stability we require that all roots of (54) have negative real part. This will occur if and only if

$$\alpha_1\beta\overset{*}{p}_{11} + \alpha_2\beta\overset{*}{p}_{22} - 2(\delta + r) < 0 \qquad (55)$$

and

$$(\alpha_1\beta\overset{*}{p}_{11} - \delta - r)(\alpha_2\beta\overset{*}{p}_{22} - \delta - r) - \alpha_1\alpha_2\beta^2\overset{*}{p}_{12}\overset{*}{p}_{21} > 0\,. \qquad (56)$$

In general it is impossible to reduce (55) and (56) to a single condition for stability of the disease-free state. We can, however, examine two special cases and develop sufficient conditions for stability and instability for the general case.

For complete assortative mixing

$$p_{ij}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{57}$$

conditions (55) and (56) become

$$\alpha_1\beta + \alpha_2\beta - 2(\delta + r) < 0 \tag{58}$$

and

$$(\alpha_1\beta - \delta - r)(\alpha_2\beta - \delta - r) > 0 \tag{59}$$

which readily reduces to

$$\frac{\alpha_1\beta}{\delta + r} < 1 \quad \text{and} \quad \frac{\alpha_2\beta}{\delta + r} < 1, \tag{60}$$

the expected result for a pair of non-interacting single subgroups.

For proportionate mixing

$$p_{ij}(t) = \begin{bmatrix} \bar{p}_1 & \bar{p}_2 \\ \bar{p}_1 & \bar{p}_2 \end{bmatrix}. \tag{61}$$

we have

$$\overset{*}{p}_{11} = \overset{*}{p}_{21} = \overset{*}{\bar{p}}_1 = \frac{\alpha_1\Lambda_1}{\alpha_1\Lambda_1 + \alpha_2\Lambda_2} \tag{62}$$

and

$$\overset{*}{p}_{21} = \overset{*}{p}_{22} = \overset{*}{\bar{p}}_2 = \frac{\alpha_2\Lambda_2}{\alpha_1\Lambda_1 + \alpha_2\Lambda_2}. \tag{63}$$

Then (55) and (56) become

$$\alpha_1\beta + \alpha_2\beta - 2(\delta + r) < 0 \tag{64}$$

and

$$(\alpha_1\beta\overset{*}{\bar{p}}_1 - \delta - r)(\alpha_2\beta\overset{*}{\bar{p}}_2 - \delta - r) - \alpha_1\alpha_2\beta^2\overset{*}{\bar{p}}_1\overset{*}{\bar{p}}_2 \;>\; 0 \tag{65}$$

respectively. These two conditions can be reduced to a single condition for stability of the disease-free state given by

$$\frac{\alpha_1\beta\overset{*}{\bar{p}}_1 + \alpha_2\beta\overset{*}{\bar{p}}_2}{\delta + r} \;<\; 1\,; \tag{66}$$

this is a weighted average of the sum of the assortative mixing conditions.

In general it is not possible to obtain a single explicit condition from (55) and (56) which will guarantee stability of the disease-free state. We have, however, established the following sufficient conditions.

I. If the system is locally asymptotically stable under complete assortative mixing then it is locally asymptotically stable under any form of mixing defined with constant ϕ_{ij} .

II. If the conditions in (60) are both violated, then the disease-free state is unstable under any form of mixing defined with constant ϕ_{ij} .

III. If one of the conditions in (60) is satisfied and the other violated then the disease-free state may be either locally asymptotically stable or unstable depending upon the particular mixing behavior of the population.

To illustrate the effect of mixing on the dynamical behavior of the system, consider the following example. Suppose that the population consists of 20,000 individuals; 2000 in subgroup 1 and 18,000 in subgroup 2. We choose this particular partition in light of the "core-group" concept which Hethcote and Yorke (REF) have shown to be so important in the spread of a sexually transmitted disease. We assume that 1% of each subgroup is infected at time $t = 0$. The transmissivity rate, β, we take to be 0.5 and let the recovery rate $r = 2.0$; finally, we take $\mu = 0.04$, $\delta = 0.1$, and $\Lambda_i = (S_i^o + I_i^o)\mu$.

We begin with a seemingly trivial, yet very important, observation: if $\alpha_1 = \alpha_2$ then mixing has no effect on the epidemic in the population as a whole, but only influences the magnitude of infection in the subgroups. This is clear by adding equation

(47) and (49) obtaining a single equation for $\dot{T}$ and equations (48) and (50) obtaining a single equation for $\dot{I}$. Thus, a size difference between subgroups does not result in any differential behavior of the model as mixing is changed.

The most obvious source of population heterogeneity is differences in sexual activity levels of various subgroups. We therefore consider an example where $\alpha_1 = 7$ and $\alpha_2 = 1$ so that the "core" group is above the endemic threshold and the non-core group is below the threshold. The average activity of the population is 1.6 which is below the endemic threshold. Under these conditions mixing can have a dramatic effect on the dynamics of the model. This can be seen in Figures 7, 8 and 9. In the case of Figure 7 there is a stable disease-free state in the "non-core" group and a stable endemic level of disease in the "core" group. This occurs when mixing is completely assortative, i.e. there is no mixing between the two subgroups. In Figure 8 we introduce a low rate of mixing between the two subgroups by ϕ to be the identity matrix. This results in a stable endemic level of disease in both subgroups. As the level of mixing between the two subgroups increases, the behavior of the model changes to a state where the disease-free equilibrium becomes stable in both subgroups. This is illustrated for the case of proportionate mixing in Figure 9. Thus, as the level of mixing between the two subgroups increases, the disease-free equilibrium of the "non-core" group switches from stable to unstable and back to stable again while the disease-free state of the "core" group is respectively unstable, unstable, and stable.

As a matter of notational convenience in what follows, we introduce the following definitions:

$$R_1 \equiv \frac{\alpha_1 \beta}{\delta + r}, \tag{67}$$

$$R_2 \equiv \frac{\alpha_2 \beta}{\delta + r}, \tag{68}$$

$$x \equiv \overset{*}{p}_{11}, \tag{69}$$

and

$$y \equiv \overset{*}{p}_{22}. \tag{70}$$

Using these, our stability criteria (55) and (56) may be written as

$$R_1 x + R_2 y < 2 \tag{71}$$

and

$$R_1 x + R_2 y < 1 + R_1 R_2 (x + y - 1). \tag{72}$$

In order to develop a better understanding of these two conditions we explore the region of stability in the R_1:R_2 plane for any fixed value of x and y. We have already established that the unit square, $R_1 < 1$ and $R_2 < 1$, is stable independent of x and y, and that $R_1 > 1$ and $R_2 > 1$ is always unstable for any x and y. Treating (71) and (72) as equalities and solving each for R_2 we obtain

$$R_2 = \frac{2 - xR_1}{y}, \tag{73}$$

and

$$R_2 = \frac{1 - xR_1}{y - (x + y - 1)R_1}. \tag{74}$$

For any fixed x and y, equation (73) defines a line in the R_1:R_2 plane with intercepts at (0,2/y) and (2/x,0). Equation (74) has intercepts at (0,1/y) and (1/x,0). Furthermore, for R_1 in [0,2/x] equations (73) and (74) have no intersection and the curve defined by (74) always lies below the curve defined by (71). Thus, the region of stability in the R_1:R_2 plane is that region which is interior to $R_1 = 0$, $R_2 = 0$, and the graph of (72) restricted to $R_1 \in [0,1/x]$. Equation (72) is a monotone decreasing function of R_1 on this interval and is concave up if $x + y < 1$, concave down if $x + y > 1$, and linear if $x + y = 1$. These general scenarios are illustrated in Figure 10. In Figure 11 we show, using three specific examples, how this region of stability changes as x and y change. Here we set $x \equiv 0.5$, and take $y = 0.25$, 0.5, and 0.75 respectively. In Figure 12 we set $x = y = 0.9$; note that the region of stability is now converging on the unit square as it should as x and y both approach 1.

Finally, we recall that x and y are not independent of one another. Using (ii), (2), and (28) we see that

$$x = 1 - \bar{p}_2\psi \tag{75}$$

and

$$y = 1 - \bar{p}_1\psi. \tag{76}$$

Thus in Figure 10, our three possible scenarios correspond to disassortative mixing ($\psi > 1$), proportionate mixing ($\psi = 1$), and assortative mixing ($\psi < 1$). Substituting (75) and (76) into (71) and (72) we obtain

$$\psi_{c1} = \frac{R_1 + R_2 - 2}{R_1 + (R_2 - R_1)\bar{p}_1}, \tag{77}$$

and

$$\psi_{c2} = \frac{1 + R_1R_2 - R_1 - R_2}{R_1R_2 - R_1 + (R_1 - R_2)\bar{p}_1} \tag{78}$$

as the boundary curves for dividing the stable and unstable regions in the range of $\psi(\bar{p}_1)$. Choosing fixed values for R_1 and R_2 we can plot $\psi_{c1}(\bar{p}_1)$ and $\psi_{c2}(\bar{p}_1)$ and identify regions of stability in the range of ψ, however, it is sufficient to consider ψ_{c2} alone as the two curves always fail to intersect for $\bar{p}_1 \in [0,1]$. Three examples are shown in Figures 13, 14, and 15 where we keep $R_1 \equiv 0.1$ and choose $R_2 = 2.0, 4.0$, and 8.0 respectively. Note that by increasing R_2 we obtain a smaller and smaller region of ψ space which is stable.

Finally, in Figure 16 we have plotted ψ_{c2} for our simulation examples shown in Figures 7, 8, and 9. At equilibria, $\bar{p}_1 = 0.4375$, and we see that increasing ψ (increasing the level of mixing between the two subgroups) takes us from the unstable to the stable region. As we increase the value of $\bar{p}_1$ at equilibria, say by making subgroup 1 larger, the level of mixing required to produce stability progressively increases as well. For $\bar{p}_1$ sufficiently large, the disease-free equilibria is always unstable.

VI. Conclusions: We have presented an intuitive sociological explanation for the general mixing representation in a single-sex population. Since all mixing functions can be written in the form of general mixing (REFS), this provides a generic explanation of the way in which mixing between subgroups can be described. As we have pointed out, however, this does not imply that mixing occurred in this way. Rather, that whatever the actual mixing was, it can be described mathematically as having taken place in this way. The general representation may be interpreted as reserving a fraction, $\phi_{ij}(t)\bar{p}_j(t)$, of each subgroup i's partnerships for interactions with subgroup j and distributing all remaining partnerships in the population at random among the various subgroups (considered now with a reduced level of sexual activity). Although not developed here, this same general interpretation also works for the case of the general representation of all solutions to the two-sex mixing problem and for the arbitrary connected case.

Specific examples of mixing solutions with parameters which are outside the unit interval were examined. Our interpretation of general mixing remains intact even in these cases. Given the importance of mixing in the modelling of both sexually transmitted and vector transmitted diseases (REFS), it is useful and important to have a "feel" for what any particular mixing function implies. With this understanding, we may be able to better interpret model predictions. It is also useful to know, that whatever complicated strategy we may like to develop for sexual mixing and partner selection, it can be reduced to a very simple formula, namely general mixing.

We have also described a hypersurface upon which the general representation reduces to proportionate mixing between subgroups. For an N-group problem, their are N(N-1)/2 interactions between distinct subgroups in the population and we have found a proportionate mixing surface between each pair of subgroups. This surface divides the mixing behavior between the two groups into two types: assortative and disassortative. Thus we examine in more detail the two-group mixing problem as this may provide insight into intergroup interactions. We characterize the biological extremes of mixing which are possible between the two groups. The "worst-case" and "best-case" mixing strategies for each subgroup, and for the population as a whole are determined. In order to show the effect of mixing on the behavior of a sexually transmitted disease epidemic we have calculated the stability of the disease-free endemic state in a general SIS/SIR model where we incorporate a general solution to the mixing problem. Specific illustrations of the role that mixing between subgroups can have are illustrated with numerical simulations of this model. Here we show how the dynamic behavior of the epidemic may change simply as a result of the mixing pattern between the subgroups changing. Using the basic reproductive numbers for the two subgroups when they are not mixing, R_1 and R_2, we have characterized the region of the R_1:R_2 plane which results in stability of the disease-free state and how this region changes with respect to changes in the equilibrium level of mixing in the population. We further describe this behavior in terms of the preference function $\psi(\bar{p}_1)$ which is a measure of the level of mixing between the two subgroups.

Despite the important role that mixing plays on the dynamics of sexually transmitted diseases, our understanding of this process is still in its infancy. Our results here, represent another step towards the clarification of some of the important aspects of the social/sexual mixing process and its role in sexually transmitted disease epidemiology.

Acknowledgements: This research has been partially supported by NSF Grant DMS-8906580, NIAID Grant R01 A129178-02, and Hatch Project Grant NYC 151-409, USDA awarded to Carlos Castillo-Chavez. Steve Blythe also acknowledges support from the Office of the Dean of the College of Agriculture and Life Sciences as well as the Mathematical Sciences Institute at Cornell University. Finally, the authors wish to thank Jane S. Huling for her assistance in typing various drafts of this manuscript.

References:

Figure Captions

Figure 1: A diagrammatic explanation of the General Mixing Representation developed by Busenberg and Castillo-Chavez for a single-sex population with multiple interacting subgroups.

Figure 2: A valid solution, in this case complete assortative mixing, of the mixing problem obtained with negative parameters in the General Mixing Representation.

Figure 3: The proportionate mixing surface between two interacting subgroups of a single-sex.

Figure 4: The possible extremes in mixing between two interacting subgroups of a single sex as represented by the preference function $\psi(\bar{p}_1)$. The preference function for any mixing solution must lie between maximal disassortative mixing and complete assortative mixing. The preference function for proportionate mixing is plotted as well, as a common reference point.

Figure 5: Examples of the behavior of the preference function for preferred mixing.

Figure 6: Examples of the behavior of the preference function for the general mixing representation.

Figure 7: For the parameter values described in the text, the number of infected individuals in subgroup 1, subgroup 2, and the total when the two-group model is simulated under complete assortative mixing, $\psi(\bar{p}_1) \equiv 0$. In this case $R_1 > 1$ and $R_2 < 1$, hence, the disease-free state of subgroup 1 is unstable and the disease-free state of subgroup 2 is stable.

Figure 8: As in Figure 7 only with $\psi(\bar{p}_1) \equiv 0.5$; now the disease-free state is unstable for both subgroups in the population.

Figure 9: As in Figure 7 and Figure 8 only with $\psi(\bar{p}_1) \equiv 1$; by further increasing the mixing between the two subgroups in the population the disease-free state becomes stable.

Figure 10: A general diagram of the region of stability for the disease-free state of the population in the R_1:R_2 plane for any fixed value of the p_{ij}'s at equilibria. If mixing between the two subgroups is disassortative, $\psi > 1$, the region of stability is concave, if mixing is assortative, $\psi < 1$, the region of stability is convex. Proportionate mixing, $\psi = 1$, divides these two cases. Note that the region of stability always includes the unit square and never includes points where $R_1 > 1$ and $R_2 > 1$.

Figure 11: A specific example of Figure 10 where x = 0.5 and y changes from 0.25 to 0.5 to 0.75.

Figure 12: Here, we have plotted the region of stability for x = y = 0.9; as x and y approach 1 this region begin to approach the unit square as expected.

Figure 13: The stable and unstable regions of the disease-free state in ψ space. Here $R_1 = 0.1$ and $R_2 = 2$.

Figure 14: As in Figure 13 only with R_2 increased to 4; the region of stability for the disease-free state has now moved up and to the right.

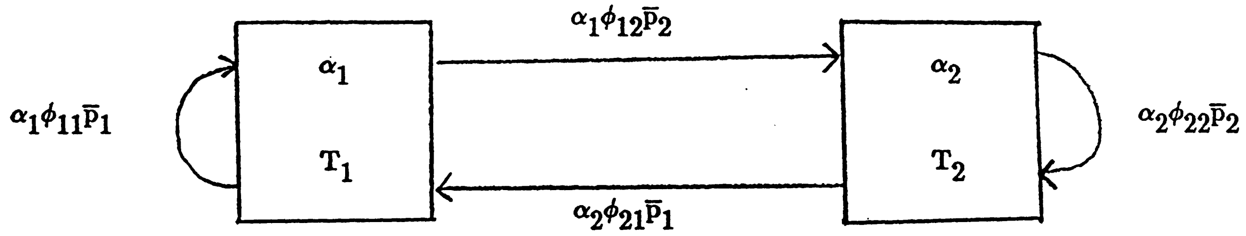
Figure 15: Again as in Figure 13 and 14 but with $R_2 = 8$.

Figure 16: The stable and unstable regions for the disease-free state in ψ space for the examples in our simulations described in the text and illustrated in Figure 7, 8, and 9. For these examples, $\bar{p}_1$ at equilibria was 0.4375 and $\psi(0.4375) > 0.787692$ results in stability of the disease-free state.

Figure 1

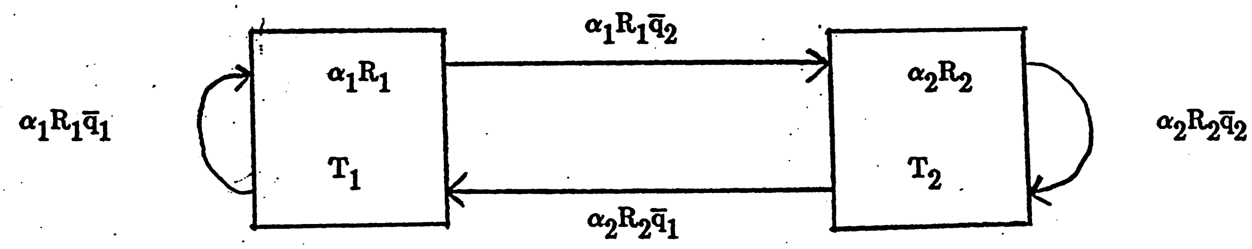
A Sociological Interpretation of General Mixing

PREFERENCE



Thus, an average individual in subgroup 1 has already formed $\alpha_1(\phi_{11}\bar{p}_1 + \phi_{12}\bar{p}_2) = \alpha_1(1 - R_1)$ sexual partnerships and must still obtain an additional $\alpha_1 R_1$ partners. Similarly, an individual in subgroup 2 has, on average, formed $\alpha_2(\phi_{21}\bar{p}_1 + \phi_{22}\bar{p}_2) = \alpha_2(1\text{-}R_2)$ partnerships and has $\alpha_2 R_2$ remaining.

PROPORTIONATE MIXING



$$\bar{q}_1 = \frac{\alpha_1 R_1 T_1}{\alpha_1 R_1 T_1 + \alpha_2 R_2 T_2} \qquad \bar{q}_2 = \frac{\alpha_2 R_2 T_2}{\alpha_1 R_1 T_1 + \alpha_2 R_2 T_2}$$

We now calculate

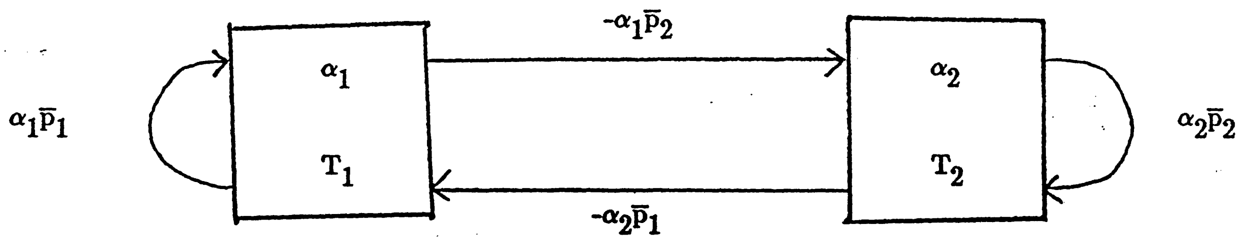
$$\begin{aligned} P_{ij} &= \phi_{ij}\bar{p}_j + R_i\bar{q}_j \\ &= \left(\phi_{ij}\bar{p}_j + \frac{R_i\alpha_j R_j T_j}{\alpha_1 R_1 T_1 + \alpha_2 R_2 T_2}\right) \\ &= \bar{p}_j\left(\phi_{ij} + \frac{R_i R_j \bar{p}_j}{R_1\bar{p}_1 + R_2\bar{p}_2}\right) \end{aligned}$$

which is the General Mixing Representation!

Figure 2

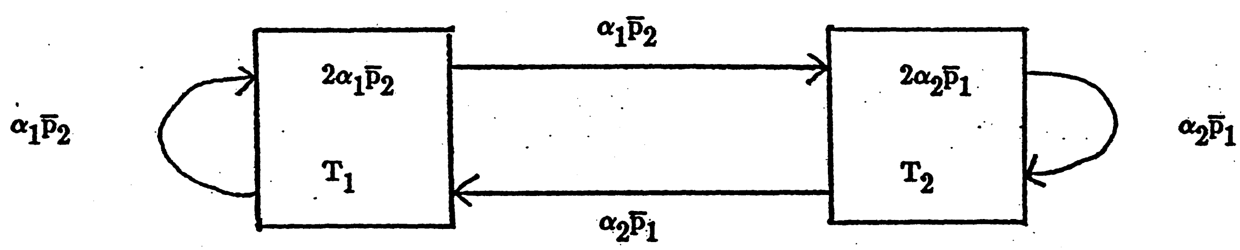
General Mixing with Negative Parameters

PREFERENCE



Thus, an average individual in subgroup 1 has already "formed" $\alpha_1(\overline{p}_1 - \overline{p}_2)$ sexual partnerships and must still obtain an additional $2\alpha_1\overline{p}_2$ partners. Similarly, an individual in subgroup 2 has, on average, "formed" $\alpha_2(\overline{p}_1 - \overline{p}_2)$ partnerships and has $2\alpha_2\overline{p}_1$ remaining.

PROPORTIONATE MIXING

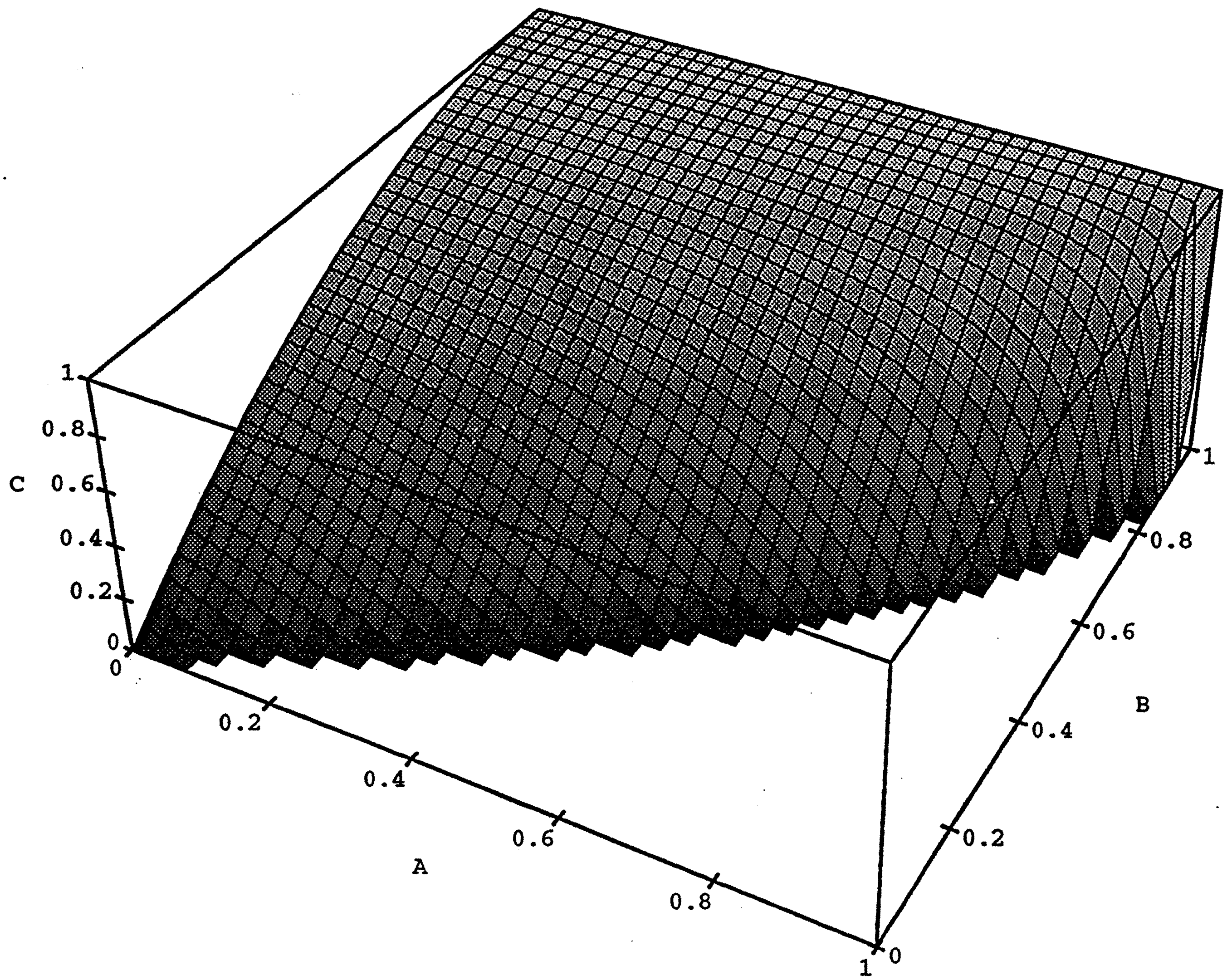


We now calculate

$$P_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

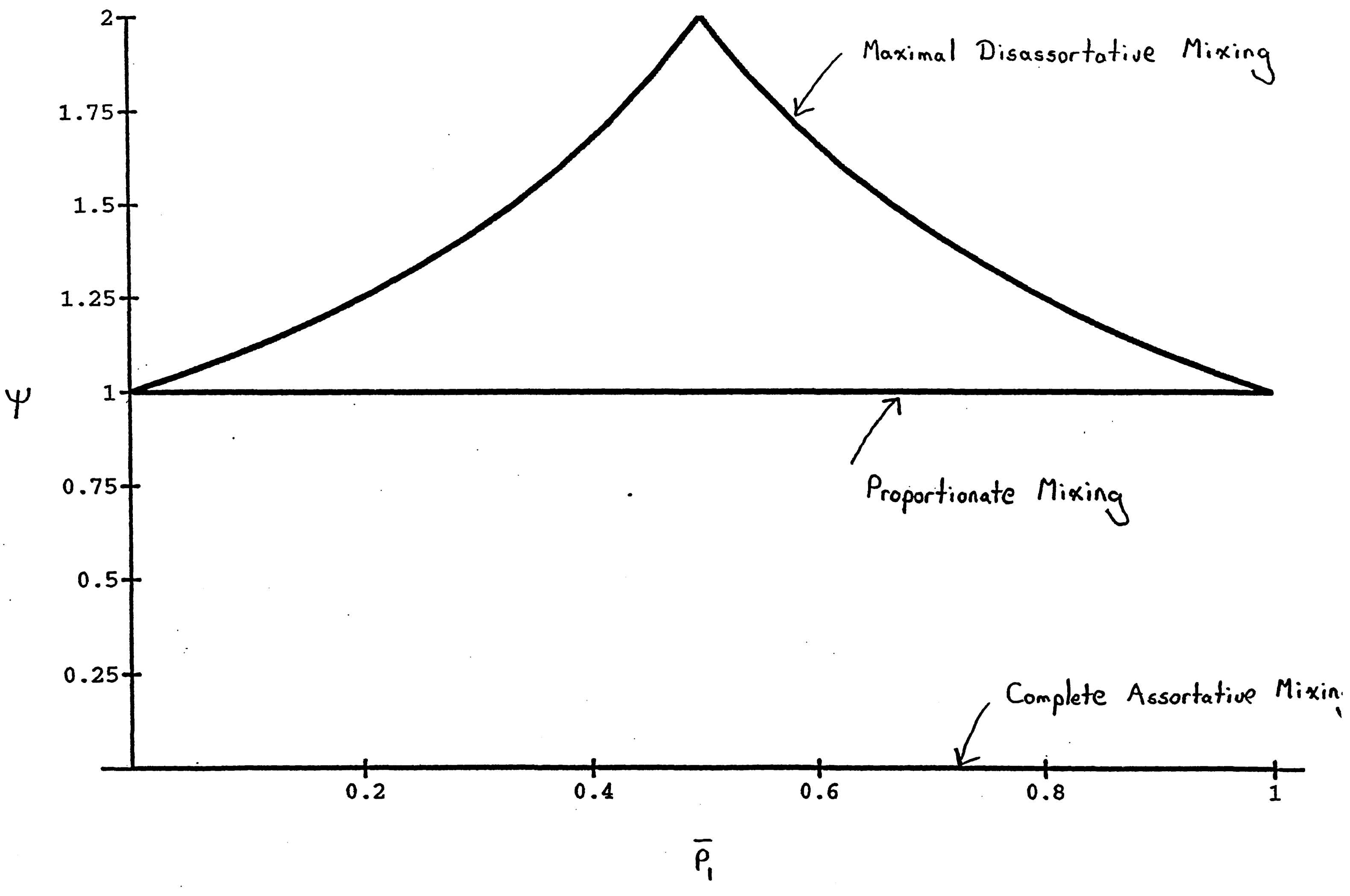
which is complete assortative mixing!

Figure 3



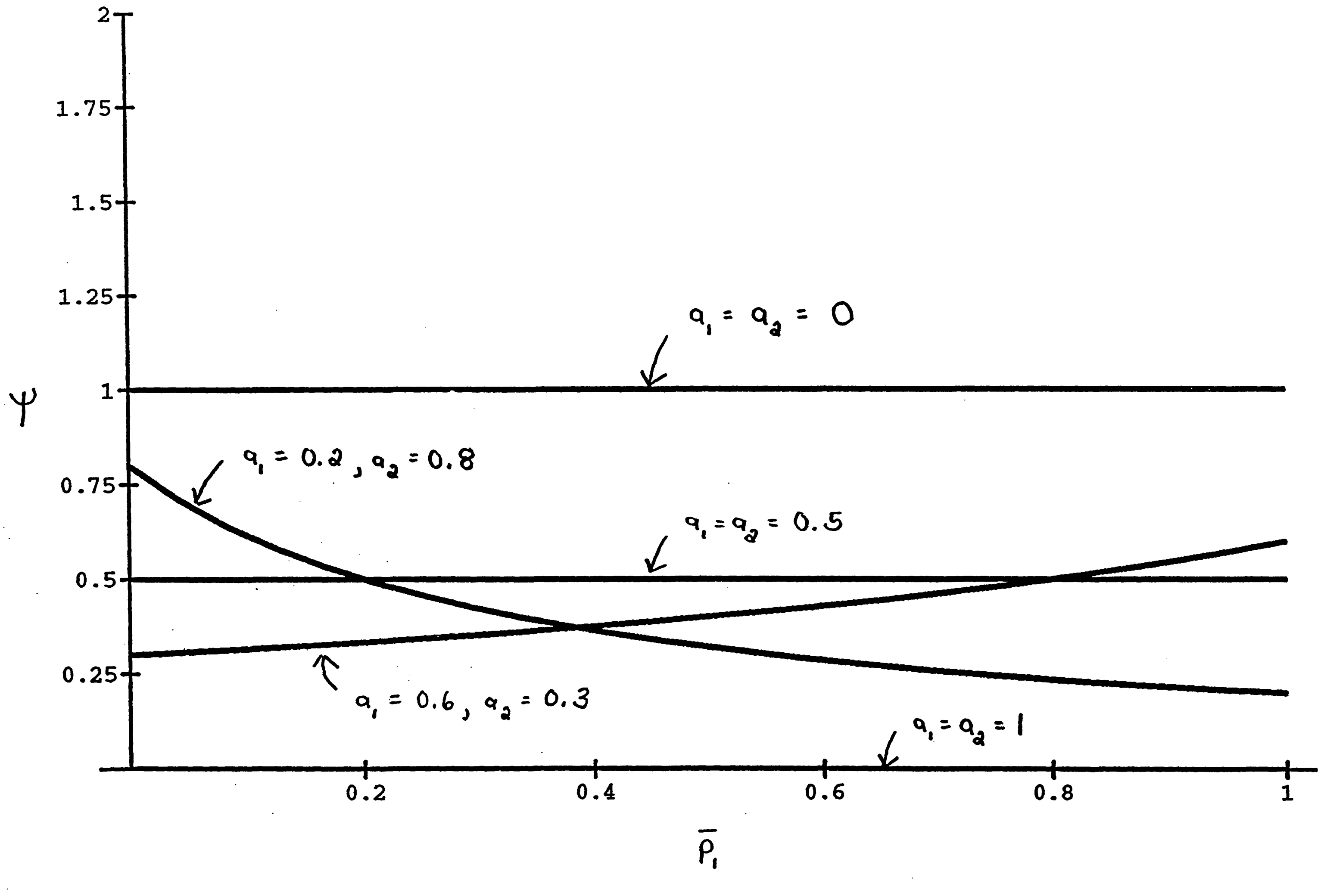
Proportionate Mixing Surface

Figure 4



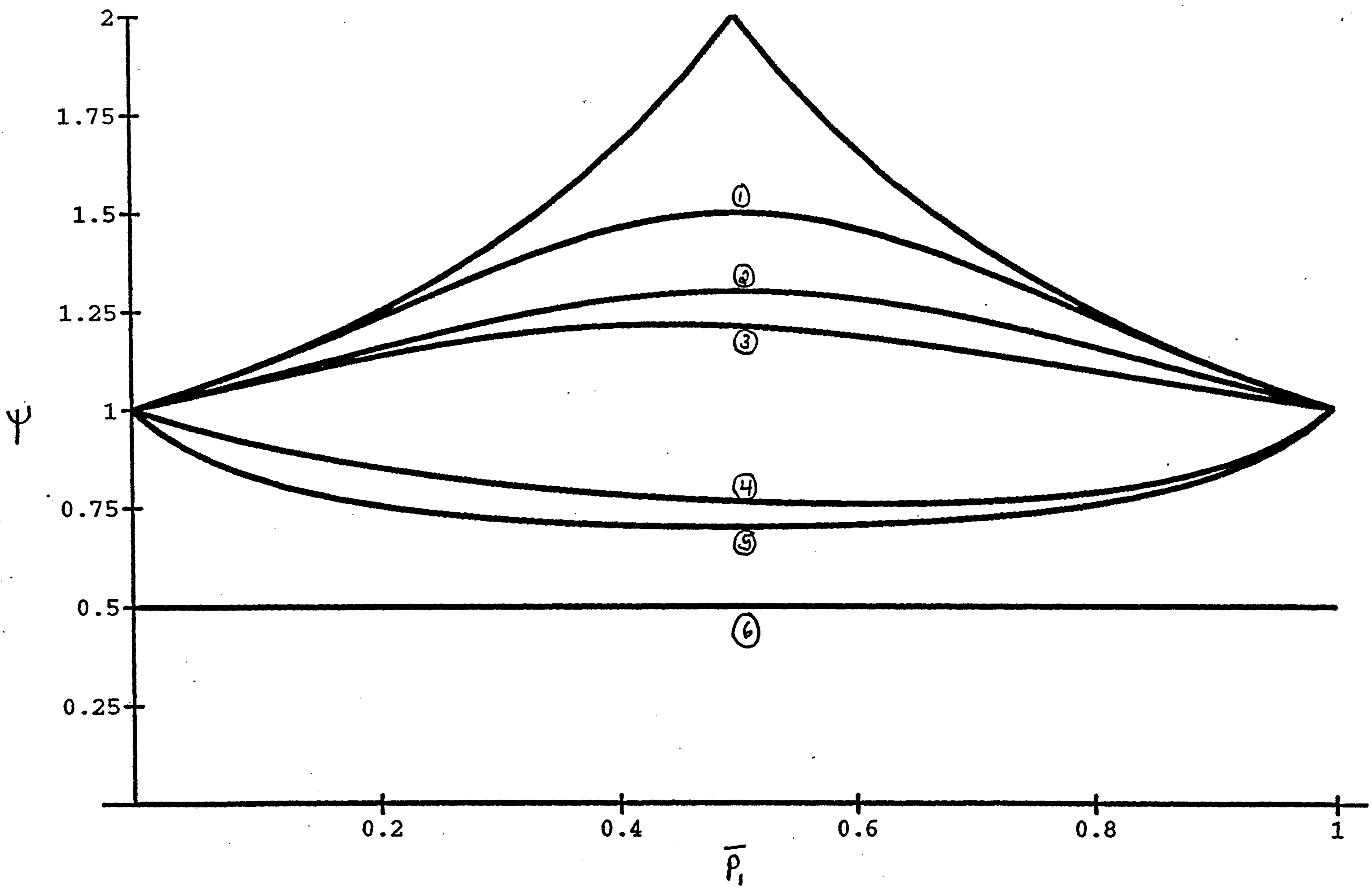
Preference Range

Figure 5



Preference for Preferred Mixing

Figure 6



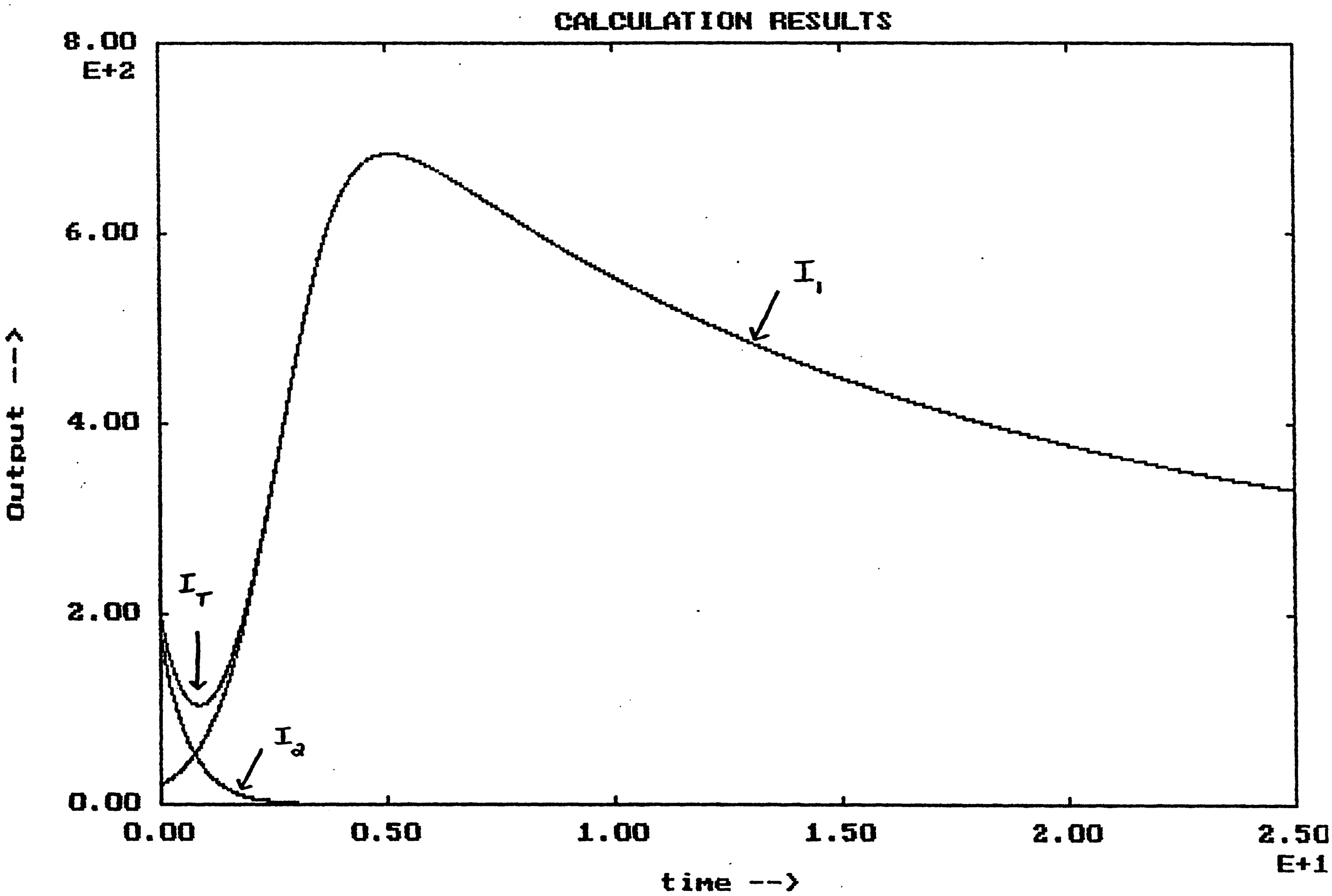
Preference for General Mixing

① $\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ④ $\phi = \begin{bmatrix} 0.8 & 0.3 \\ 0.3 & 0.5 \end{bmatrix}$

② $\phi = \begin{bmatrix} 0.3 & 0.8 \\ 0.8 & 0.3 \end{bmatrix}$ ⑤ $\phi = \begin{bmatrix} 0.8 & 0.3 \\ 0.3 & 0.8 \end{bmatrix}$

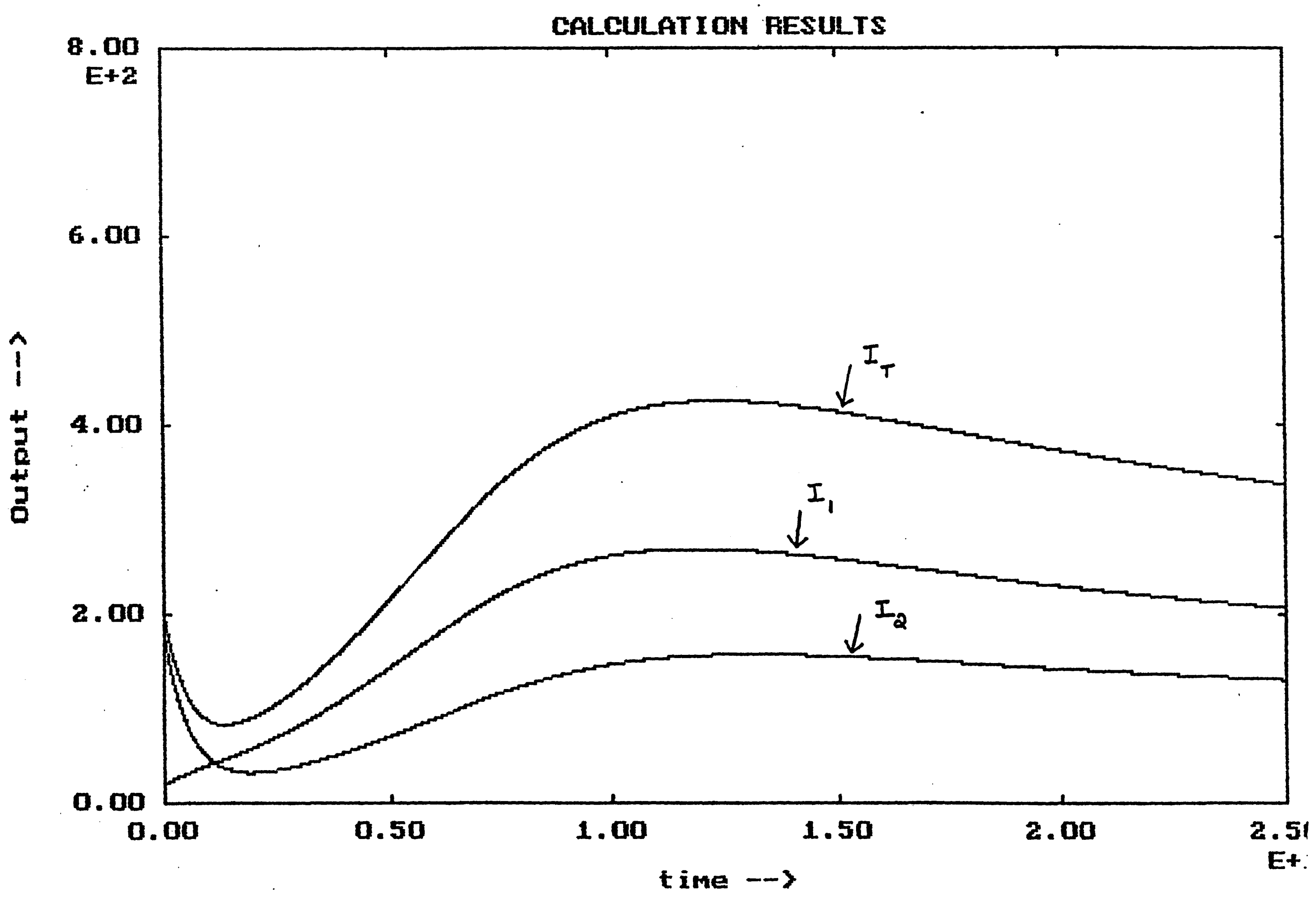
③ $\phi = \begin{bmatrix} 0.3 & 0.8 \\ 0.8 & 0.5 \end{bmatrix}$ ⑥ $\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Figure 7



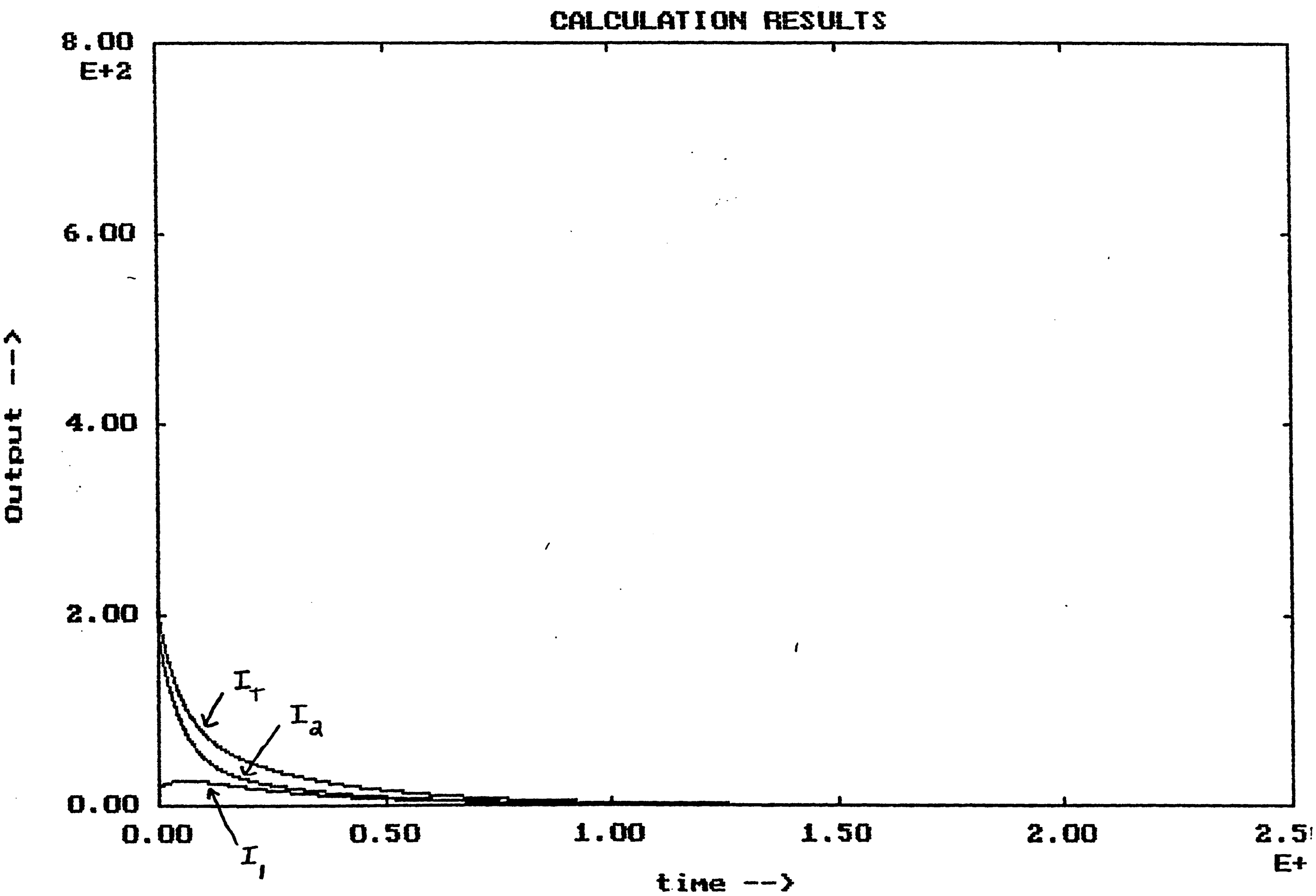
$$\phi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Figure 8



$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Figure 9



$$\Phi = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Figure 10

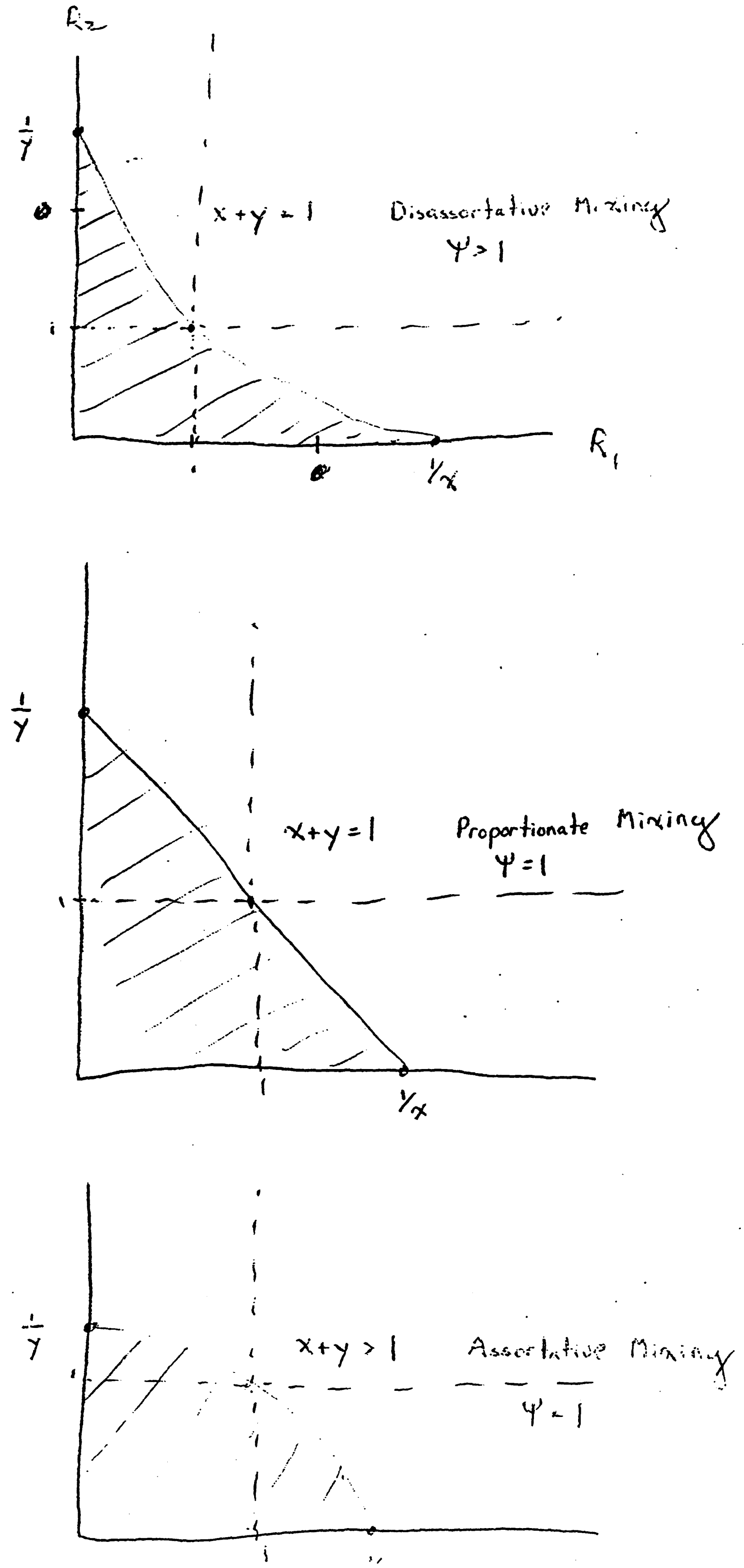
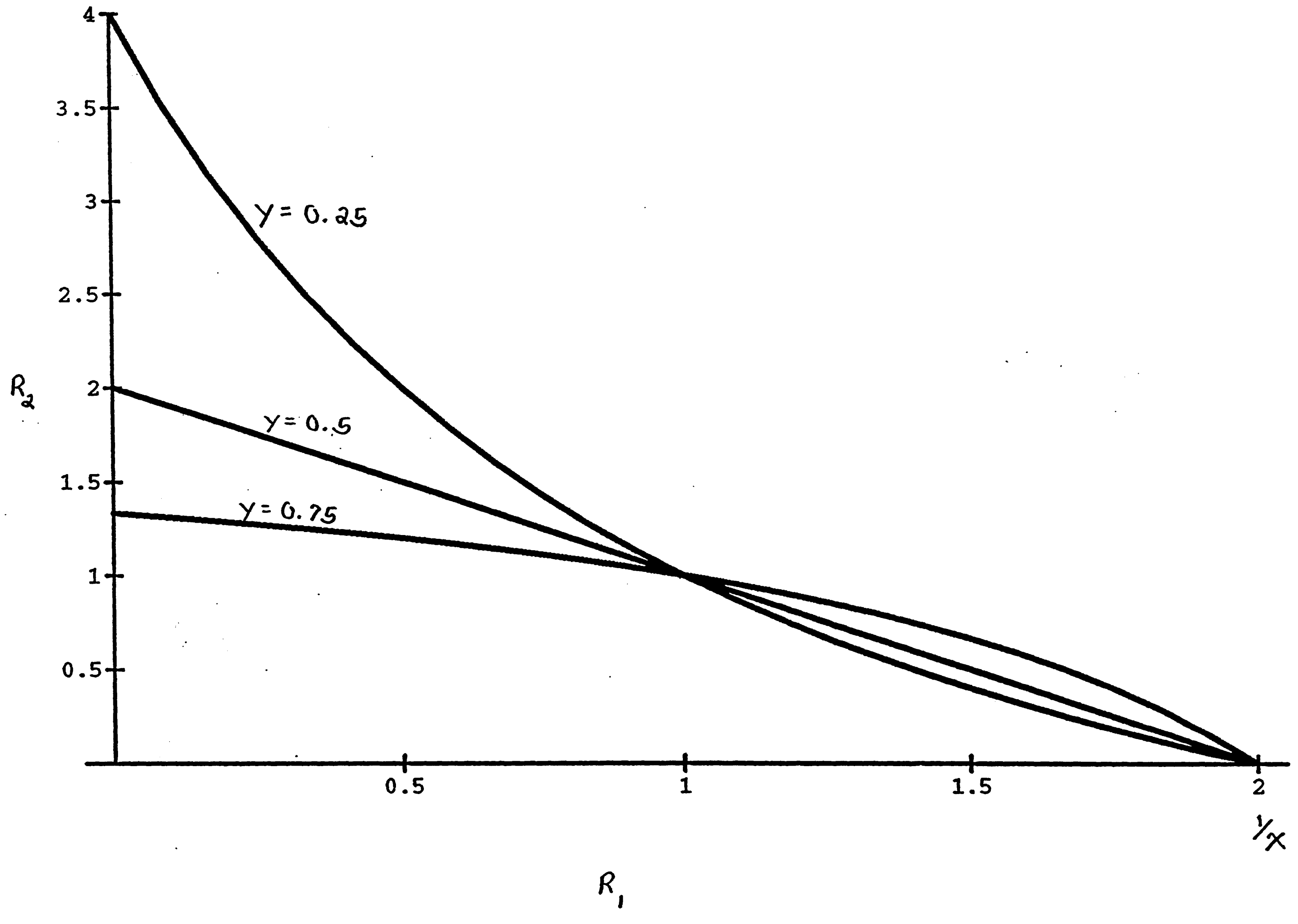

R_2
$\frac{1}{y}$
1
$x+y=1$
Disassortative Mixing
$\psi>1$
R_1
$1/x$
$\frac{1}{y}$
$x+y=1$
Proportionate Mixing
$\psi=1$
$1/x$
$\frac{1}{y}$
$x+y>1$
Assortative Mixing
$\psi<1$

Figure 11



$x \equiv 0.5$

Figure 12

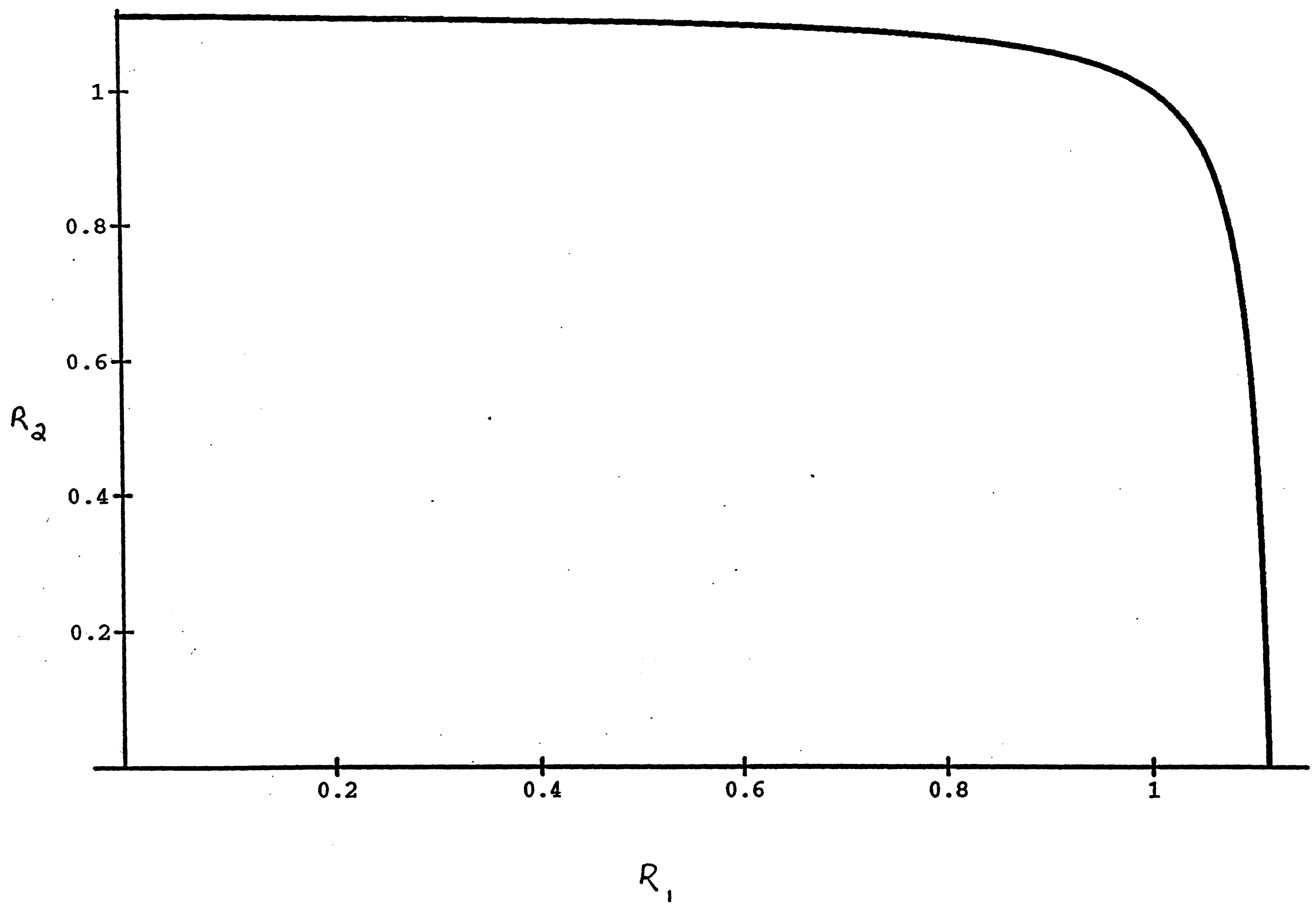


Figure 13

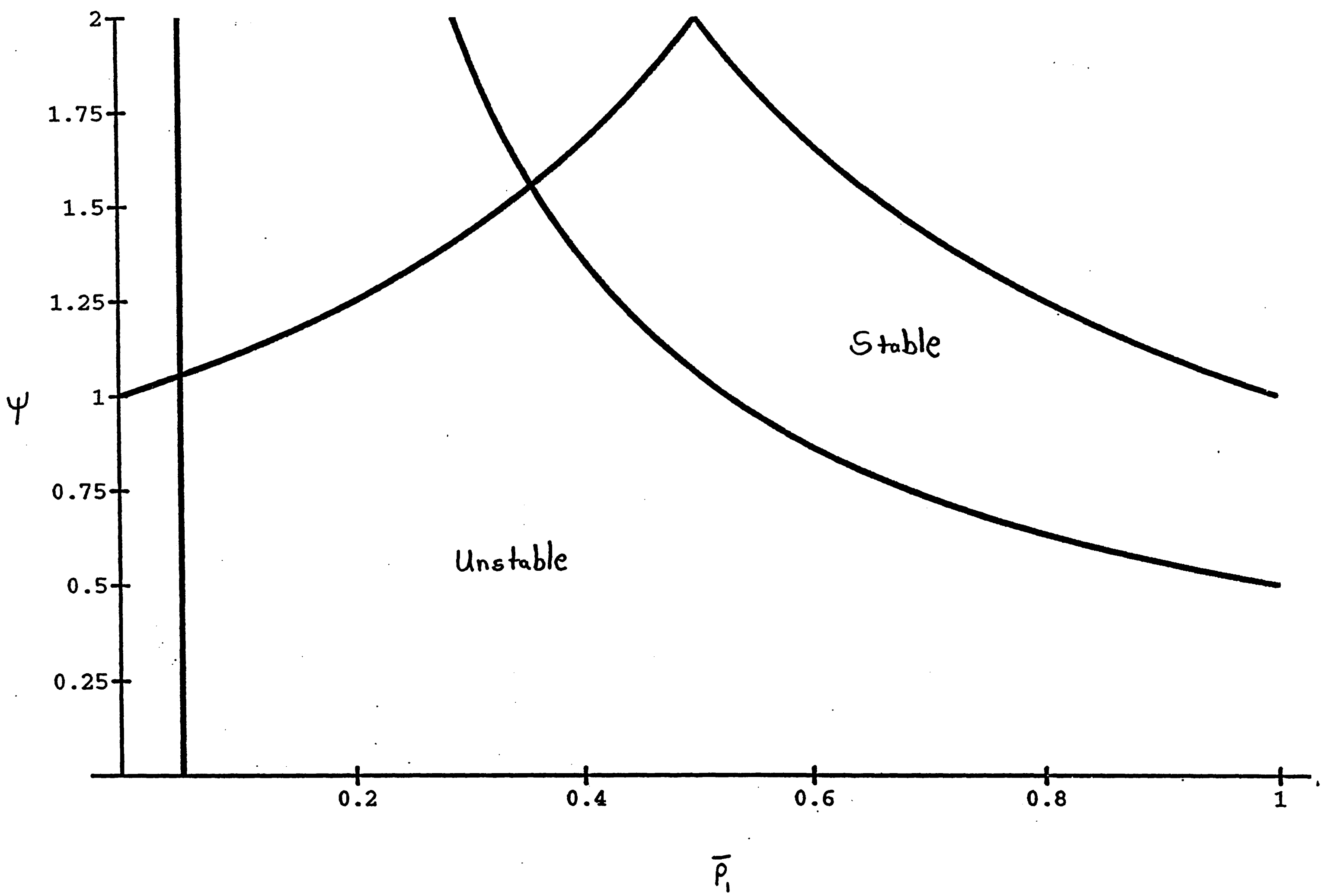


Figure 14

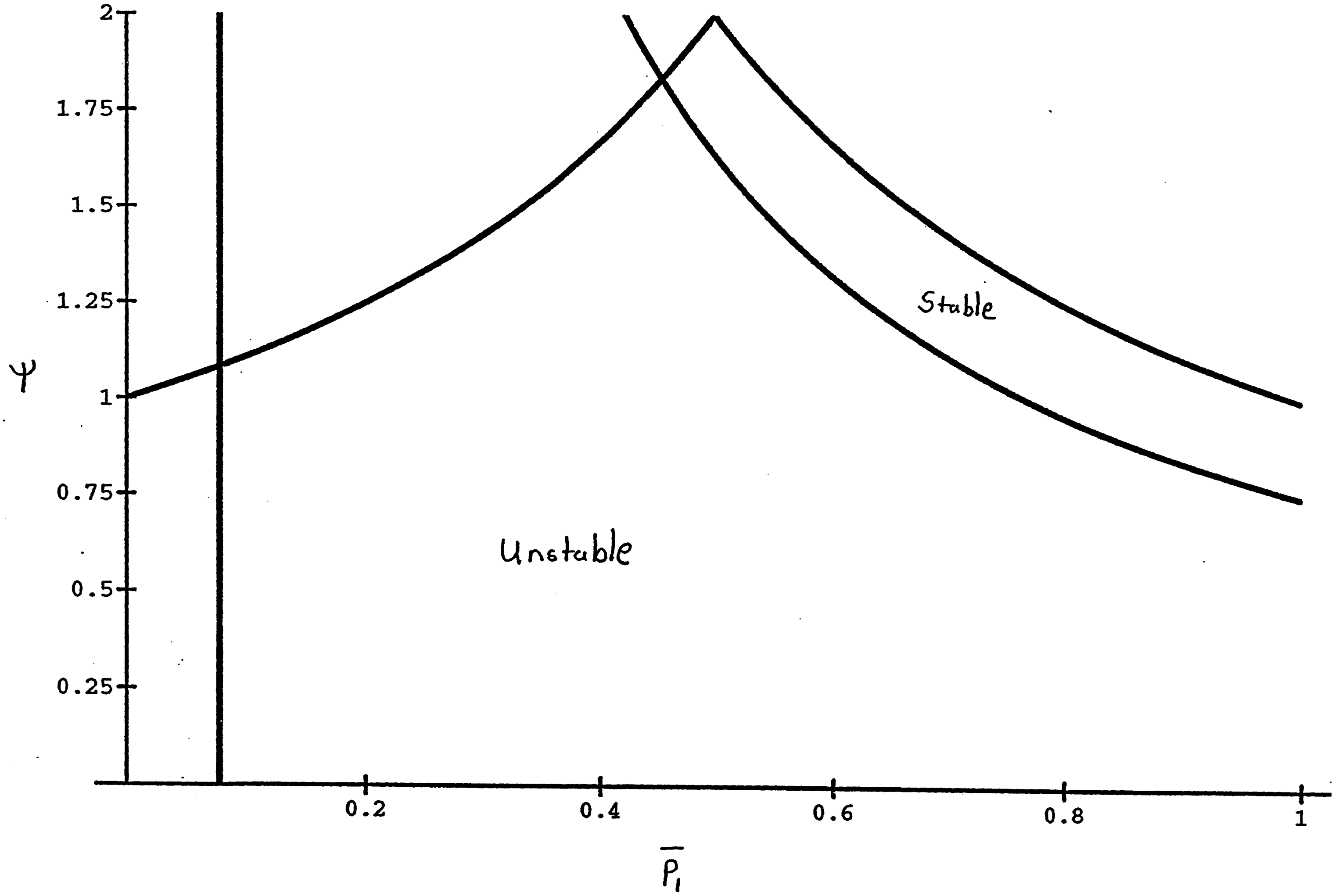

2
1.75
1.5
1.25
1
0.75
0.5
0.25
Ψ
Stable
Unstable
0.2
0.4
0.6
0.8
1
$\overline{P_1}$

Figure 15

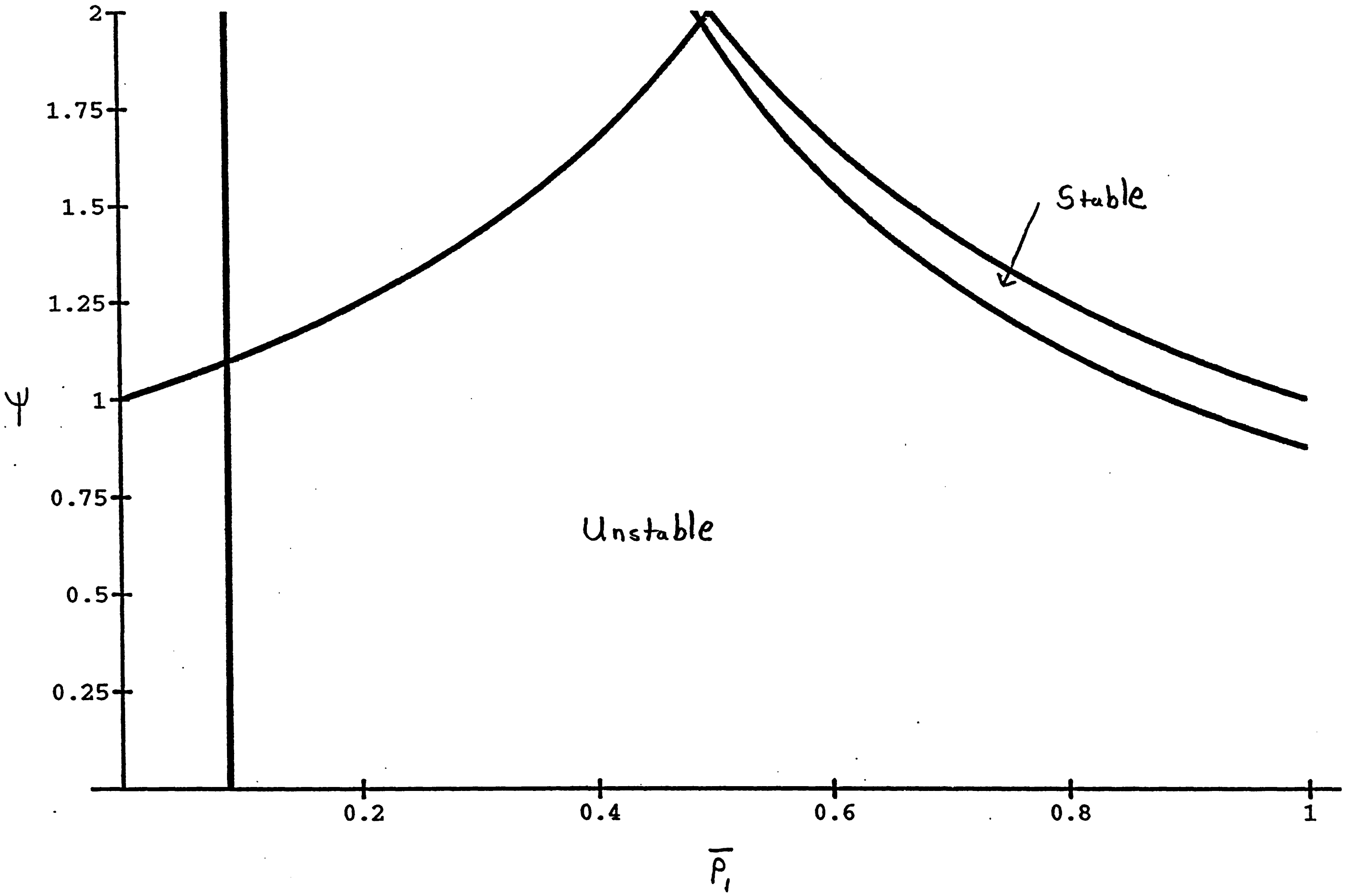
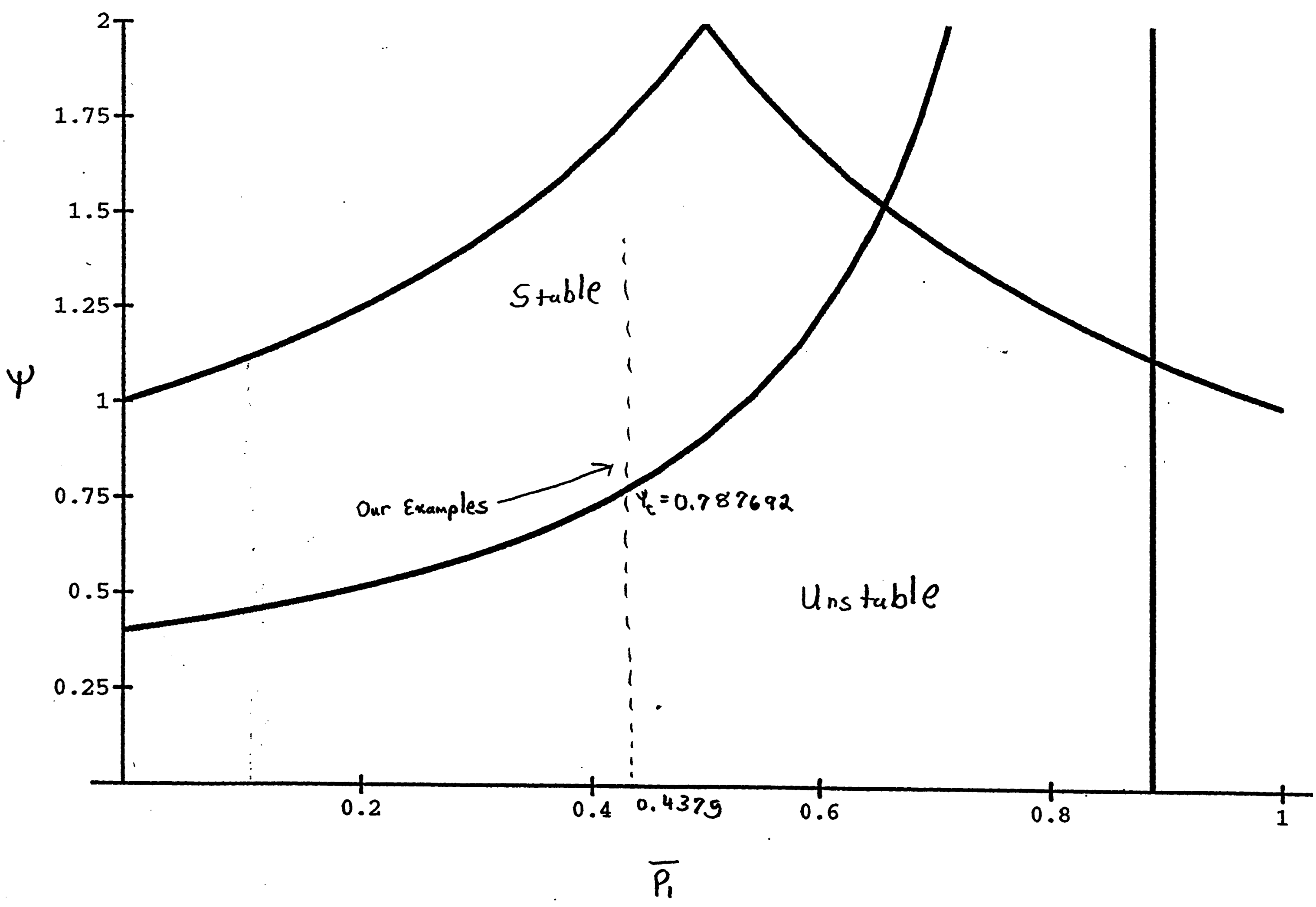


Figure 16



2
1.75
1.5
1.25
1
0.75
0.5
0.25
Ψ
Stable
Our Examples
Ψc=0.787692
Unstable
0.2
0.4
0.4379
0.6
0.8
1
P̄1